O‘ZBEKISTON RESPUBLIKASI OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

ALFRAGANUS UNIVERSITY NODAVLAT TA’LIM TASHKILOTI

SHAMSIYEV RAHIM NAJMIDDINOVICH

DIFFERENSIAL VA INTEGRAL HISOB

DARSLIK

II QISM

60610500 - Kompyuter injiniringi, 60610600 - Dasturiy injiniring, 60612100 - Kiberxafvsizlik injiniringi bakalavr ta’lim yo‘nalishlarining talabalariga tavsiya qilinadi

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Mazkur darslik oliy ta’lim muassasalarining oliy matematikani turli hajmlarda o‘rganuvchi barcha texnika yo‘nalishlari bakalavrlari uchun mo‘ljallangan.

Darslik kompyuter, kiberxavfsizlik injineringi, dasturiy injinering va sun’iy intellekt yo‘nalishlarining “Hisob” fani dasturi asosida yozilgan bo‘lib, I qismda fanning matematik analizga kirish, bir o‘zgaruvchili funksiyaning differensial va integral hisobi, ularning tadbiqlari va II qismda ikki o‘zgaruvchili funksiyaning differensial va integral hisobi, ularning tadbiqlari, maydonlar nazariyasi, qatorlar nazariyasi va kompleks o‘zgaruvchili funksiyalar nazariyasi mavzulari bayon qilingan. Paragraflarning har birida mavzuga oid asosiy tushunchalar keltirilgan, tegishli teoremalar isboti bilan berilgan va unga doir namunaviy misollar yechimlari bilan berilgan. Mavzuga oid mustaqil yechish uchun misollar ham berilgan.

Taqrizchilar:

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Yurtimizning ta’lim tizimidagi keng ko‘lamli islohatlar zamonaviy, jahon talablariga javob beradigan yuqori malakali, raqobatbardosh kadrlar tayyorlash muammolari bilan uzviy bog‘langanligi barchamizga ma’lum. Bu islohatlar o‘z navbatida oliy maktab professor-o‘qituvchilari oldiga ta’lim maqsadi va mazmunini jahon andozalariga moslashtirishdek ma’suliyatli vazifalarni yuklaydi. Bu vazifalar orasida bo‘lajak mutaxassislarning fundamental fanlarni chuqur egallashiga doir qat’iy talablarini qondira oladigan yangi o‘quv adabiyotlarini yaratish dolzarb masalasi yotadi.

Ana shu dolzarb masalani hal etishda oz bo‘lsada o‘z hissasini qo‘shish maqsadida muallif tomonidan I.A.Karimov nomidagi Toshkent Davlat texnika universiteti, Alfraganus universiteti talabalariga bir qancha yillar davomida o‘qigan ma’ruzalari va olib borgan amaliy mashg‘ulotlari asosida to‘plangan tajribalarga tayangan holda differensial va integral hisob kursidan darslik yozishga jazm etdi.

Kitobxon e’tiboriga havola etilayotgan mazkur “Differensial va integral hisob” darsligi oliy o‘quv yurtlarining texnika yo‘nalishlari talabalari uchun mo‘ljallangan. Unda keltirilgan mavzular oliy o‘quv yurtlarining barcha texnika yo‘nalishlari uchun tasdiqlangan namunaviy fan dasturiga to‘liq mos keladi.

Darslikdagi ayrim teoremalar isbotsiz keltirilgan. Teoremalarning ana shu isbotlari bilan qiziquvchi kitobxon ularni adabiyotlar ro‘yxatidagi asosiy adabiyotlardan topishi mumkin.

Kitobda mavzular, ta’riflar, teoremalar, misollar va rasmlarning tartib raqamlari har bir bob uchun alohida qo‘yilgan. Teorema isbotining, hamda misol va masala yechimining boshiga ►, oxiriga esa ◄ belgi qo‘yilgan.

Muallif mazkur darslikni yozishni fizika-matematika fanlari nomzodi, dotsent marhum G‘.Shodmonov tavsiya qilganini va qo‘l yozma yozilish jarayonida u o‘zining qimmatbaho maslahatlarini ayamaganligini chuqur hurmat bilan e’tirof etadi.

Muallif kitob haqida bildirilgan har qanday fikr-mulohazalarni kitobning sifatini yaxshilash uchun ko‘rsatilgan yordam sifatida minnatdorchilik bilan qabul qiladi (shamsievrahim@rambler.ru).

Muallif.

VII BOB. KO‘P O‘ZGARUVCHILI FUNKSIYALAR

Bir o‘zgaruvchili funksiyalar tabiatda mavjud bo‘lgan barcha bog‘liqliklarni ochib bera olmaydi. Bunga bir necha misol keltiraylik:

7.1-Misol. Tomonlarining uzunliklari x va y bo‘lgan to‘g‘ri to‘rt burchakning yuzi

$$S = xy$$

formula bilan hisoblanadi. Har bir x va y juftlikka yuzning aniq bir S qiymati mos keladi, ya’ni S ikki o‘zgaruvchining funksiyasi.

7.2-Misol. Qirralarining uzunliklari x, y, z bo‘lgan parallelepiped hajmi

$$V = xyz$$

formula bilan hisoblanadi. Shuning uchun V uchta o‘zgaruvchining funksiyasi.

7.3-Misol. Yer atmosferasi nuqtasidagi γ namlik darajasi uning joylashgan o‘rni: $x -$kenglik, $y -$uzoqlik, $z -$balandlikka va nuqtadagi t temperaturaga bog‘liq, ya’ni γ to‘rt o‘zgaruvchining funksiyasi.

Bunday bog‘liqliklarni o‘rganish uchun ko‘p o‘zgaruvchili funksiya tushunchasi kiritiladi. Biz ko‘p o‘zgaruvchili funksiyalarni o‘rganishni ikki o‘zgaruvchili funksiyalarni o‘rganish bilan chegaralanamiz, chunki ko‘p o‘zgaruvchili funksiyalarga xos bo‘lgan muhim xususiyatlar ikki o‘zgaruvchili funksiyalardayoq namoyon bo‘ladi. Bu xususiyatlar ko‘p o‘zgaruvchili funksiyalarga hech qanday qiyinchiliksiz umumlashtiriladi. Bundan tashqari ikki o‘zgaruvchili funksiyalarni va ularning xossalarini geometrik talqin qilish mumkin.

7.1. Ikki o‘zgaruvchili funksiya tushunchasi

Ochiq va yopiq to‘plamlar. $x, y \in \mathbb{R}$ haqiqiy sonlardan tashkil topgan tartiblangan (x, y) juftliklar to‘plami $\mathbb{R}$ haqiqiy sonlar to‘plamining dekart kvadrati deb ataladi va $\mathbb{R}^2$ ko‘rinishda belgilanadi (agar x, y sonlar juftligida ulardan qaysinisi birinchi, qaysinisi ikkinchi ekanligi ko‘rsatilgan bo‘lsa, bu juftlik tartiblangan deyiladi, (x, y) juftlikda $x -$birinchi, $y -$ikkinchi son). $\mathbb{R}^2$ to‘plamni dekart koordinatalar tekisligi, (x, y) juftlikni esa bu tekislikning nuqtasi deb qarash mumkin. Bu tekislikda $M(x_1, y_1)$ va $N(x_2, y_2)$ nuqtalar orasidagi masofa sifatida

$$\rho(M, N) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

miqdorni olgan edik.

Xuddi shu singari $x, y, z \in R$ haqiqiy sonlardan tashkil topgan (x, y, z) tartiblangan sonlar to‘plamidan $\mathbb{R}^3$ dekart fazoni hosil qilamiz va bu yerda $M(x_1, y_1, z_1)$ va $N(x_2, y_2, z_2)$ nuqtalar orasidagi masofa

$$\rho(M, N) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

tenglik bilan aniqlanadi.

7.1-Ta’rif. $M_0(x_0, y_0)$ nuqtaning ε atrofi deb

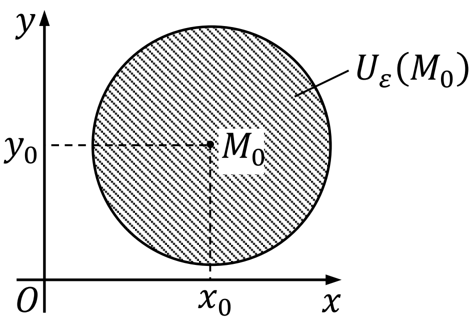
$$U_\varepsilon(M_0) = \left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon\right\}$$

to‘plamga aytiladi. Ko‘rinib turibdiki bu to‘plam markazi $M_0(x_0, y_0)$ nuqtada va radiusi ε bo‘lgan doiradan iborat (7.1-rasm).

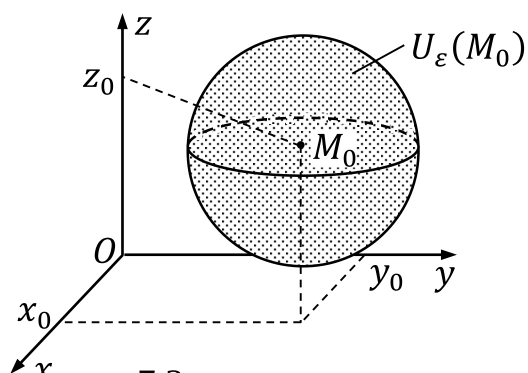
$\mathbb{R}^3$ fazodagi $M_0(x_0, y_0, z_0)$ nuqtaning ε atrofi markazi M_0 nuqtada va radiusi ε bo‘lgan

$$U_\varepsilon(M_0) = \left\{(x, y, z): \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \varepsilon\right\}$$

shardan iborat bo‘ladi (7.2-rasm).



7.1-rasm



7.2-rasm

7.2-Ta’rif. Agar $D \subset \mathbb{R}^2$ to‘plam M nuqtasining D to‘plamda to‘liq yotadigan $U_\varepsilon(M)$ atrofi (ε atrofi) mavjud bo‘lsa, M nuqta D *to‘plamning ichki nuqtasi* deyiladi.

7.3-Ta’rif. Agar D to‘plamning barcha nuqtalari ichki nuqtalardan iborat bo‘lsa, D to‘plam *ochiq to‘plam* deb ataladi.

7.4-Misol. $\mathbb{R}^2$ tekislikdagi eng sodda ochiq to‘plam bu nuqtalarning ε −atrofi bo‘ladi.

►Haqiqattan ham, ixtiyoriy $M \in \mathbb{R}^2$ nuqta va uning ε −atrofini, ya’ni $U_\varepsilon(M)$ to‘plamni qaraymiz. Agar $N \in U_\varepsilon(M)$ bo‘lsa, 7.1-Ta’rifga ko‘ra $\rho(M, N) < \varepsilon$ bo‘ladi. $\varepsilon_1 = \varepsilon - \rho(M, N)$ musbat sonni olamiz. Agar P nuqta N nuqtaning ε_1 −atrofiga tegishli bo‘lsa, $\rho(N, P) < \varepsilon_1$ bo‘ladi. Uchburchak tengsizligiga ko‘ra

$$\rho(M, P) \leq \rho(N, P) + \rho(N, M) < \varepsilon_1 + \rho(M, N) = \varepsilon.$$

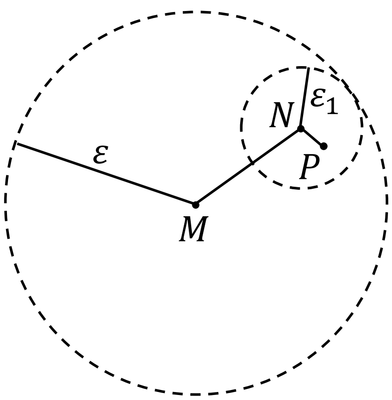
Demak P nuqta M nuqtaning $\varepsilon -$atrofi $U_\varepsilon(M)$ ga tegishli ekan. $P \in U_{\varepsilon_1}(N)$ nuqta ixtiyoriy ravishda tanlangan edi, shuning uchun $U_{\varepsilon_1}(N) \subset U_\varepsilon(M)$ deb xulosa chiqarishimiz mumkin.

Shunday qilib, ixtiyoriy $N \in U_\varepsilon(M)$ nuqta to‘liq $U_\varepsilon(M)$ to‘plamda yotuvchi $U_{\varepsilon_1}(N)$ atrofga ega ekan. Bu esa N nuqtaning ichki nuqta va demak $U_\varepsilon(M)$ to‘plamning ochiq to‘plam ekanligidan darak beradi. 7.3-rasmda hozir keltirilgan isbotning geometrik talqini berilgan.◄

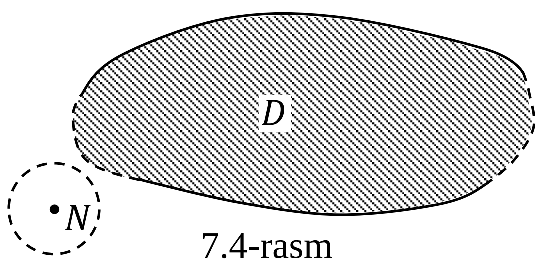
7.4-Ta’rif. M nuqtaning $U_\varepsilon(M)$ atrofidan M nuqtani olib tashlanishidan hosil bo‘lgan $\dot{U}_\varepsilon(M) = U_\varepsilon(M) \backslash \{M\}$ to‘plamga M nuqtaning o‘zi kirmagan atrofi deb ataladi.

Agar M nuqtaning atrofini olishda $\varepsilon > o$ qiymatini ko‘rsatish shart bo‘lmasa biz umumiylashtirib M nuqtaning biror atrofi deb aytamiz va uni $U(M)$ (nuqtaning o‘zi kirmagan atrof uchun esa $\dot{U}(M)$) orqali belgilaymiz.

7.5-Ta’rif. Agar $M \in \mathbb{R}^2$ nuqtaning ixtiyoriy ε atrofida D to‘plamga tegishli bo‘lgan ham, tegishli bo‘lmagan ham nuqtalar yotsa, M nuqta D to‘plamning *chegara nuqtasi* deb ataladi. D to‘plamning barcha chegara nuqtalari to‘plami uning *chegarasi* deb ataladi va u ∂D orqali belgilanadi.



7.3-rasm



7.4-rasm

7.5-Misol. M_0 nuqta ε atrofining chegarasi markazi M_0 nuqtada va radiusi ε bo‘lgan aylanadan iborat:

$$\partial U_\varepsilon(M_0) = \left\{(x, y) \colon \sqrt{(x - x_0)^2 + (y - y_0)^2} = \varepsilon\right\}.$$

7.6-Ta’rif. Agar shunday musbat r soni topilib, $O(0,0)$ nuqtaning $r -$atrofida $D \subset \mathbb{R}^2$ to‘plam to‘liq yotsa, D to‘plam chegaralangan to‘plam deb ataladi.

7.7-Ta’rif. Chegara nuqtalarini (chegarasini) ham o‘zida saqlagan to‘plam yopiq to‘plam deb ataladi va ular $\overline{D}$ ko‘rinishda belgilanadi.

7.8-Ta’rif. Agar N$\in \mathbb{R}^2$ nuqtaning shunday ε atrofi topilib, bu atrof D to‘plam bilan kesishmasa, N nuqta D to‘plamning tashqi nuqtasi deb ataladi (7.4-rasm).

7.9-Ta’rif. Agar D to‘plamning ixtiyoriy ikki nuqtasini bu to‘plamga tegishli bo‘lgan uzluksiz chiziq bilan tutashtirish mumkin bo‘lsa, u *bog‘lamli to‘plam* deb ataladi.

Masalan, doira, ellipsning ichki qsmi, to‘g‘ri to‘rtburchakning ichki qismi bog‘lamli to‘plam bo‘ladi. Umumiy nuqtaga ega bo‘lmagan ukkita doira bo‘glamli to‘plam bo‘lmaydi.

7.10-Ta’rif. Bog‘lamli ochiq to‘plam soha deb ataladi.

Ikki o‘zgaruvchili funksiya tushunchasi. Ikki o‘zgaruvchili funksiya tushunchasi ham bir o‘zgaruvchili funksiya singari kiritiladi.

7.11-Ta’rif. (x, y) tartiblangan sonlar juftligining $D \in \mathbb{R}^2$ to‘plami va $E \in \mathbb{R}$ sonli to‘plam berilgan bo‘lsin. Har bir $(x, y) \in D$ juftlikka yagona $z \in E$ sonni mos qo‘yuvchi f moslikka D to‘plamda aniqlangan va qiymatlari E to‘plamda bo‘lgan *ikki o‘zgaruvchili funksiya* deb ataladi va $z = f(x, y)$ ko‘rinishda belgilanadi. D to‘plam funksiyaning *aniqlanish*, E to‘plam esa *qiymatlari sohasi* deb ataladi.

Ikki o‘zgaruvchili funksiya yana $z = z(x, y)$, $z = \varphi(x, y)$ va hokazo ko‘rinishda ham belgilanadi.

Har bir (x, y) tartiblangan sonlar juftligiga to‘g‘ri burchakli koordinatalar sistemasida tekislikning yagona M nuqtasi mos keladi va aksincha har bir M nuqtaga yagona tartiblangan (x, y) juftlik mos keladi. Shuning uchun ikki o‘zgaruvchili funksiyani M nuqtaning funksiyasi sifatida qarash mumkin va ba’zan $z = f(x, y)$ o‘rniga $z = f(M)$ belgilashdan foydalaniladi. Biz bundan keyin ana shu ikki belgilashdan foydalanamiz.

Ikki o‘zgaruvchili funksiyalar bir o‘zgaruvchili funksiyalar singari turli usullar bilan berilishi mumkin. Biz misollarda asosan funksiyalar formulalar yordamida beriladigan analitik usuldan foydalanamiz. Bunday hollarda funksiyaning aniqlanish sohasi formulalar ma’noga ega bo‘ladigan nuqtalar to‘plami hisoblanadi.

Ikki o‘zgaruvchi funksiyalarga bir nechta misol qaraymiz.

7.6-Misol. $z = 1 + x^2 + y^2$ funksiya ixtiyoriy (x, y) juftlikda aniqlangan, shuning uchun uning aniqlanish sohasi butun Oxy tekislikdan iborat bo‘ladi: $D = \{(x, y): (x, y) \in \mathbb{R}^2\}$, qiymatlar sohasi esa $E = [1, +\infty)$ bo‘ladi.

7.7-Misol. $z=\sqrt{1-x^2-y^2}$ funksiya kvadrat ildiz ostidagi ifoda noldan kichik bo‘lmaydigan, ya’ni $1-x^2-y^2\geq 0$ yoki $x^2+y^2\leq 1$ bo‘ladigan nuqtalarda aniqlangan. Shuning uchun uning aniqlanish sohasi markazi koordinatalar boshida va radiusi 1 ga teng bo‘lgan doiradan iborat: $D=\{(x,y): x^2+y^2\leq 1\}$, qiymatlar sohasi [0,1] kesmadan iborat: $E=[0,1]$.

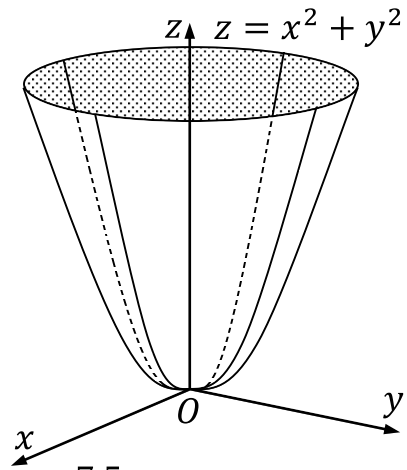
7.8-Misol. $z=1/\sqrt{x^2+y^2-1}$ funksiya $x^2+y^2-1>0$ yoki $x^2+y^2>1$ bo‘ladigan nuqtalar to‘plamida aniqlangan. Shuning uchun bu funksiyaning aniqlanish sohasi markazi koordinatalar boshida va radiusi birga teng bo‘lgan doiraning tashqi qismidan iborat bo‘ladi: $D=\{(x,y): x^2+y^2>1\}$, qiymatlar sohasi esa $E=(0,+\infty)$ bo‘ladi.

Qaralga misollardan ko‘rinib turibdiki, ikki o‘zgaruvchili funksiyaning aniqlanish sohasi butun Oxy tekislik ham yoki uning ma’lum bir qismi ham bo‘lishi mumkin ekan.

Ikki o‘zgaruvchili funksiyalarni geometrik tasvirlash. Ma’lumki, bir o‘zgaruvchili funksiya tekislikda $y=f(x)$ tenglama bilan aniqlanadigan chiziq ko‘rinishida tasvirlanadi. Ikki o‘zgaruvchili funksiya esa fazoda $z=f(x,y)$ tenglama bilan aniqlanadigan sirt ko‘rinishida rasvirlanadi.

$z=f(x,y)$ funksiya Oxy tekislikning biror D sohasida aniqlangan bo‘lsin. U holda har bir $(x,y)\in D$ nuqtaga uch o‘lchovli fazoning $\big(x,y,f(x,y)\big)$ nuqtasi mos keladi. Bunday $\big(x,y,f(x,y)\big)$ nuqtalar to‘plamiga $z=f(x,y)$ *funksiyaning grafigi* deb ataymiz. Masalan, $z=4x+3y+2$ funksiyaning grafigi $4x+3y-z+2=0$ tekislik bo‘ladi.

$z=x^2+y^2$ funksiyaning grafigi aylanma paraboloid bo‘ladi (7.5-rasm).



7.5-rasm

7.2. Ikki o‘zgaruvchili funksiyaning limiti

$f(x,y)$ funksiya $M_0(x_0,y_0)$ nuqtaning biror $U(M_0)$ atrofida aniqlangan bo‘lsin (M_0 nuqtaning o‘zida aniqlanmagan ham bo‘lishi mumkin).

7.12-Ta’rif. Ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta>0$ son topilib, $M_0(x_0,y_0)$ nuqtadan farqli va $\rho(M_0,M)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M(x,y)\in U_\delta(M_0)$ nuqtalar uchun

$$|f(M) - A| < \varepsilon \qquad (7.1)$$

tengsizlik o‘rinli bo‘lsa, A soni $f(M)$ funksiyaning $M_0(x_0, y_0)$ nuqtadagi limiti deb ataladi va $A = \lim_{M \to M_0} f(M)$ yoki $A = \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y)$ ko‘rinishda yoziladi.

Ta’rifdan agar limit mavjud bo‘lsa, M nuqta M_0 nuqtaga qanday yo‘nalish bilan intilishiga (bunday yo‘nalishlar soni cheksiz ko‘p, bir o‘zgaruvchili funksiyalarda esa $x \to x_0$ intilish faqat ikkita yo‘nalish bo‘yicha: chapdan va o‘ngdan) bog‘liq emasligi kelib chiqadi).

Ikki o‘zgaruvchili funksiya limitining geometrik talqini quyidagicha. $\varepsilon > 0$ soni qanday bo‘lishidan qat’iy nazar, $M_0(x_0, y_0)$ nuqtaning shunday δ atrofi topiladiki, bu atrofdan olingan $M_0(x_0, y_0)$ nuqtadan farq qiluvchi barcha $M(x, y)$ nuqtalarga mos keluvchi $z = f(x, y)$ sirt nuqtasi applikatasining A sonidan modul bo‘yicha farqi ε sonidan kichik bo‘ladi.

7.9-Misol. $\lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2 - y^2}{x^2 + y^2}$ limitni hisoblang.

► $M(x, y)$ nuqtani $O(0,0)$ nuqtaga $y = kx$ to‘g‘ri chiziq bo‘ylab intiltiramiz, bu yerda k −o‘zgarmas son. U holda

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - k^2x^2}{x^2 + k^2x^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2(1 - k^2)}{x^2(1 + k^2)} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1 - k^2}{1 + k^2} = \frac{1 - k^2}{1 + k^2}.$$

$z = \dfrac{x^2 - y^2}{x^2 + y^2}$ funksiya $O(0,0)$ nuqtada limitga ega emas, chunki k koeffisiyentning turli qiymatlarida funksiyaning limiti bir xil emas.◄

7.13-Ta’rif. Agar shunday P o‘zgarmas son topilib, $D \subset \boldsymbol{R}^2$ sohaga tegishli bo‘lgan barcha $M(x, y) \in D$ nuqtalarda $|f(M)| \leq P$ tengsizlik o‘rinli bo‘lsa, $z = f(x, y)$ funksiya D sohada chegaralangan deyiladi.

7.14-Ta’rif. Agar $M_0(x_0, y_0)$ nuqtaning biror $U(M_0)$ atrofida $f(M)$ funksiya chegaralangan bo‘lsa, $f(M)$ funksiya M_0 nuqta atrofida chegaralangan deyiladi.

7.1-Teorema. Agar $\lim_{M \to M_0} f(M) = A$ va $A -$chekli son bo‘lsa, $f(M)$ funksiya M_0 nuqta atrofida chegaralangan bo‘ladi.

► $\lim_{M \to M_0} f(M) = A$ tenglikdan ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilib, $M_0(x_0, y_0)$ nuqtadan farqli va $\rho(M_0, M) < \delta$ tengsizlikni qanoatlantiruvchi barcha $M(x,y) \in U_\delta(M_0)$ nuqtalar uchun $|f(M) - A| < \varepsilon$ tengsizlikning bajarilishi kelib chiqadi. Shuning uchun

$$|f(M)| = |f(M) - A + A| \leq |f(M) - A| + |A| < \varepsilon + |A|.$$

Bu esa $f(M)$ funksiya M_0 nuqta atrofida chegaralanganligini anglatadi.◄

7.15-Ta'rif. Agar $\lim\limits_{M \to M_0} f(M) = 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada cheksiz kichik deb ataladi.

Agar $z = f(M)$ funksiya M_0 nuqtada chekli A limitga ega bo'lsa, $\alpha(M) = f(M) - A$ funksiya M_0 nuqtada cheksiz kichik bo'ladi. Haqiqattan ham, $\lim\limits_{M \to M_0} f(M) = A$ bo'lganligidan, ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilib, $M_0(x_0, y_0)$ nuqtadan farqli va $\rho(M_0, M) < \delta$ tengsizlikni qanoatlantiruvchi barcha $M(x,y) \in U_\delta(M_0)$ nuqtalar uchun $|\alpha(M)| = |f(M) - A| < \varepsilon$ tengsizlik o'rinli bo'lishi kelib chiqadi. Bu esa $\lim\limits_{M \to M_0} \alpha(M) = 0$ ekanligini anglatadi.

Shunday qilib, M_0 nuqtada A soniga teng limitga ega bo'lgan funksiya uchun maxsus $f(M) = A + \alpha(M)$ tasvirni hosil qildik, bu yerda $\lim\limits_{M \to M_0} \alpha(M) = 0$.

7.2-Teorema. Agar $f(M)$ va $g(M)$ funksiyalar M_0 nuqtada mos ravishda A va B limitlarga ega bo'lsa, ularning $f(M) + g(M)$ yig'indisi, $f(M) - g(M)$ ayirmasi, $f(M) \cdot g(M)$ ko'paytmasi va $B \neq 0$ bo'lganda $f(M)/g(M)$ nisbati ham M_0 nuqtada limitga ega bo'ladi va

$$\lim_{M \to M_0} [f(M) \pm g(M)] = A \pm B, \quad \lim_{M \to M_0} [f(M) \cdot g(M)] = A \cdot B,$$

$$\lim_{M \to M_0} \frac{f(M)}{g(M)} = \frac{A}{B} \quad (B \neq 0)$$

tengliklar o'rinli bo'ladi.

►Isbotni $f(M)/g(M)$ nisbat uchun keltiramiz. Qolganlari ham xuddi shu singari isbotalanadi.

$\lim\limits_{M \to M_0} f(M) = A$, $\lim\limits_{M \to M_0} g(M) = B$ ekanligidan $f(M)$ va $g(M)$ funksiyalarni

$$f(M) = A + \alpha(M), \qquad g(M) = B + \beta(M)$$

ko'rinishda tasvirlash mumkin, bu yerda $\lim\limits_{M \to M_0} \alpha(M) = \lim\limits_{M \to M_0} \beta(M) = 0$. U holda $B \neq 0$ shartni inobatga olsak

$$\frac{f(M)}{g(M)} = \frac{A+\alpha(M)}{B+\beta(M)} =$$

$$= \frac{\frac{1}{B}[B+\beta(M)][A+\alpha(M)] - \frac{A}{B}\beta(M) - \frac{1}{B}\beta(M)\alpha(M)}{B+\beta(M)} =$$

$$= \frac{1}{B}[A+\alpha(M)] - \frac{A}{B[B+\beta(M)]}\beta(M) - \beta(M)\frac{\alpha(M)}{B[B+\beta(M)]} =$$

$$= \frac{A}{B} + \alpha(M)\frac{1}{B} - \beta(M)\frac{A}{B[B+\beta(M)]} - \beta(M)\frac{\alpha(M)}{B[B+\beta(M)]}.$$

M_0 nuqtada $(M)\frac{1}{B}$, $\beta(M)\frac{A}{B[B+\beta(M)]}$, $\beta(M)\frac{\alpha(M)}{B[B+\beta(M)]}$ hadlar cheksiz kichik funksyalar bo‘lganligi uchun (cheksiz kichik funksya bilan chegaralangan funksyaning ko‘paytmasi sifatida) ularning yig‘indisi ham cheksiz kichik bo‘ladi.

Shunday qilib $\frac{f(M)}{g(M)}$ nisbat $\frac{A}{B}$ o‘zgarmas bilan M_0 nuqtada cheksiz kichik bo‘lgan funksyaning yig‘indisi shaklida tasvirlanadi. Shuning uchun $\frac{f(x)}{g(x)}$ funksya M_0 nuqtada $\frac{A}{B}$ nisbatga teng bo‘lgan limitga ega bo‘ladi.◄

7.3. Ikki o‘zgaruvchili funksiyaning uzluksizligi

Ikki o‘zgaruvchili funksiyaning uzluksizligi tushunchasi limit tushunchasiga tayangan holda kiritiladi.

7.16-Ta’rif. Agar

i) $f(x,y)$ funksiya $M_0(x_0,y_0)$ nuqtada va uning biror atrofida aniqlangan;

ii) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$ limit mavjud;

iii) bu limit $f(x,y)$ funksiyaning M_0 nuqtadagi qiymatiga teng, ya’ni

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0) \qquad \left(\lim_{M\to M_0} f(M) = f(M_0)\right) \tag{7.2}$$

tenglik o‘rinli bo‘lsa, $z=f(x,y)$ funksiya $M_0(x_0,y_0)$ *nuqtada uzluksiz* deyiladi.

Agar funksiya biror sohaning barcha nuqtalarida uzluksiz bo‘lsa, funksiya bu *sohada uzluksiz* deyiladi. Funksiya uzluksiz bo‘lmagan nuqtalarni (funksiyaning nuqtadagi uzluksizlik shartlaridan birortasi bajarilmagan) uning *uzilish nuqtalari* deb ataladi. Funksiyaning uzilish nuqtalari butun bir *uzilish chiziqlarini* hosil qilishi ham mumkin. Masalan, $z = 1/(x^2 - y^2)$ funksiya ikkita $y = x$ va $y = -x$ uzilish chiziqlariga ega.

$\Delta z = f(x, y) - f(x_0, y_0)$ tenglik bilan aniqlanadigan miqdorga $z = f(x, y)$ funksiyaning $M_0(x_0, y_0)$nuqtadagi *to‘liq orttirmasi* deb ataladi. x va y *argumentlarning orttirmalari* esa mos ravishda $\Delta x = x - x_0$, $\Delta y = y - y_0$ formulalar orqali hisoblanadi. Bu belgilashlardan foylanaib Δz to‘liq orttirma uchun

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

formulani hosil qilamiz.

Uzluksizlikning yuqoridagi ta’rifiga teng kuchli bo‘lgan boshqa ta’rifini beramiz:

7.17-Ta’rif. Agar $z = f(x, y)$ funksiyaning $M_0(x_0, y_0)$ nuqtadagi to‘liq orttirmasi $(x, y) \to (x_0, y_0)$ da cheksiz kichik, ya’ni

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0$$

tenglik o‘rinli bo‘lsa, $f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada uzluksiz deyiladi.

7.10-Misol. $z = x^2 y$ funksiyani uzluksizlikka tekshiring.

►Funksiyaning $M(x, y)$ nuqtadagi to‘liq orttirmasini topamiz:

$$\Delta z = (x + \Delta x)^2 (y + \Delta y) - x^2 y =$$
$$= 2xy\Delta x + y(\Delta x)^2 + x^2 \Delta y + 2x\Delta x\Delta y + (\Delta x)^2 \Delta y.$$

U holda $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = 0$ ekanligi ko‘rinib turibdi. Bu esa 7.17-Ta’rifga ko‘ra $z = x^2 y$ funksiyaning ixtiyoriy $M(x, y)$ nuqtada uzluksiz ekanligini anglatadi.◄

Uzluksiz ikki o‘zgaruvchili funksiyalar quyidagi xossalarga ega:

1) Agar $z = f(x, y)$ funksiya chegaralangan yopiq sohada uzluksiz bo‘lsa, u bu sohada chegaralangan, ya’ni shunday P o‘zgarmas son topilib, sohaning ixtiyoriy nuqtasi uchun $|f(x, y)| \leq P$ tengsizlik o‘rinli bo‘ladi.

2) Agar $z = f(x, y)$ funksiya chegaralangan yopiq sohada uzluksiz bo‘lsa, u bu sohada o‘zining aniq quyi va yuqori chegaralariga erishadi.

3) Agar $z = f(x, y)$ funksiya biror D sohada uzluksiz bo‘lsa, u o‘zining ixtiyoriy ikki qiymati orasidagi barcha qiymatlarni qabul qiladi, ya’ni A va B

bu funksiyaning D sohadagi ikkita qiymati va $A < C < B$ bo‘lsa, bu sohada $f(x_0, y_0) = C$ tenglikni qanoatlantiradigan $M_0(x_0, y_0)$ nuqta mavjud.

4) Yopiq $\overline{D}$ sohada uzluksiz bo‘lgan $f(x, y)$ funksiya bu sohada chegaralangan va o‘zining eng katta va eng kichik qiymatlariga erishadi.

7.4. Ikki o‘zgaruvchili funksiyaning xususiy hosilalari

$z = f(x, y)$ funksiya $M(x, y)$ nuqtaning biror atrofida aniqlangan bo‘lsin. M nuqtada x o‘zgaruvchiga Δx orttirma beramiz, bunda y o‘zgaruvchining qiymatini o‘zgarmasdan qoldiramiz, ya’ni tekislikning $M(x, y)$ nuqtasidan $M_1(x + \Delta x, y)$ nuqtasiga o‘tamiz. Δx orttirmaning qiymatini shunday tanlaymizki, bunda M_1 nuqta M nuqtaning yuqorida aytib o‘tilgan atrofida yotsin. U holda funksiyaning

$$\Delta_x z = f(x + \Delta x, y) - f(x, y)$$

mos orttirmasini funksiyaning $M(x, y)$ nuqtadagi *x o‘zgaruvchi bo‘yicha xususiy orttirmasi* deb ataymiz.

y o‘zgaruvchi bo‘yicha xususiy orttirma xuddi shu singari aniqlanadi:

$$\Delta_y z = f(x, y + \Delta y) - f(x, y).$$

7.18-Ta’rif. Agar

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \quad \left(\lim_{\Delta x \to 0} \frac{\Delta_y z}{\Delta y} \right)$$

limit mavjud bo‘lsa, bu limit $z = f(x, y)$ funksiyaning $M(x, y)$ nuqtadagi x o‘zgaruvchi (y o‘zgaruvchi) bo‘yicha xususiy hosilasi deb ataladi va u

$$z'_x, f'_x, \frac{\partial z}{\partial x}, \frac{\partial f}{\partial x} \quad \left(z'_y, f'_y, \frac{\partial z}{\partial y}, \frac{\partial f}{\partial y} \right)$$

orqali belgilanadi.

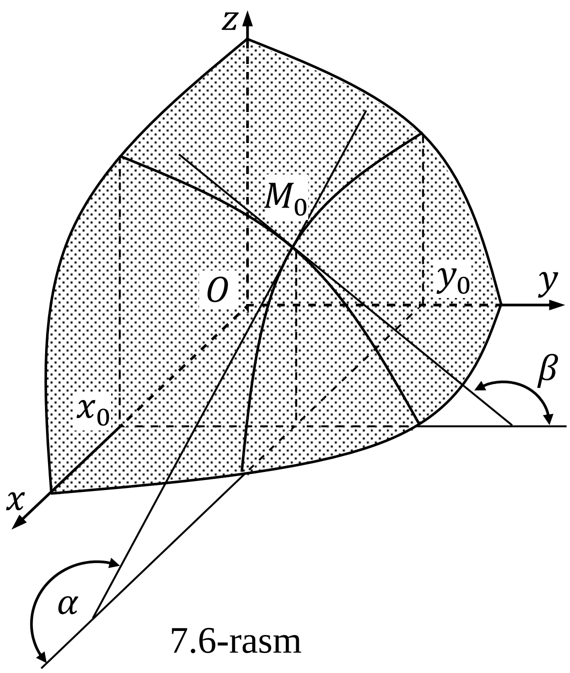
Shunday qilib, ta’rifga ko‘ra ikki o‘zgaruvchili funksiyaning x o‘zgaruvchi bo‘yicha xususiy hosilasi y o‘zgaruvchining o‘zgarmas qiymatida x ning bir o‘zgaruvchili funksiyadan olingan oddiy hosiladan iborat ekan. Shuning uchun xususiy hosilalar bir o‘zgaruvchili funksiyaning hosilasini hisoblash formulalri va qoidalariga ko‘ra hisoblanadi (bunda mos ravishda x yoki y o‘zgarmas miqdor deb hisoblanadi).

7.11-Misol. $z = xy + \sin(x - y) + 5$ funksiyaning xususiy hosilalarini toping.

► $z'_x = (xy + \sin(x - y) + 5)'_x = (xy)'_x + (\sin(x - y))'_x + (5)'_x =$

$$= y + \cos(x - y) \cdot (x - y)'_x + 0 = y + \cos(x - y).$$

$$z'_y = (xy + \sin(x-y) + 5)'_y = (xy)'_y + (\sin(x-y))'_y + (5)'_y =$$
$$= x + \cos(x-y) \cdot (x-y)'_y + 0 = y - \cos(x-y).◄$$



7.6-rasm

Ikki o‘zgaruvchili funksiya xususiy hosilalarining geometrik ma’nosi. $z = f(x,y)$ funksiyaning grafigi sirt ekanligini yuqorida ko‘rdik (7.6-rasm). $z = f(x, y_0)$ funksiyaning grafigi esa bu sirt bilan $y = y_0$ tekislik kesishishidan hosil bo‘lgan chiziqdan iborat. Bir o‘zgaruvchili funksiya hosilasining geometrik ma’nosidan kelib chiqib (3.1 ga qarang), $f'_x(x_0, y_0) = \operatorname{tg}\alpha$ degan xulosaga kelamiz, bu yerda $\alpha -$ abssissalar o‘qi bilan $z = f(x, y_0)$ funksiya grafigiga $M_0(x_0, y_0, f(x_0, y_0))$ nuqtada o‘tkazilgan urinma orasidagi burchak (7.6-rasm).

Xuddi shu singari, $f'_x(x_0, y_0) = \operatorname{tg}\beta$ bo‘ladi va bu yerda $\beta -$ordinatalar o‘qi bilan $z = f(x_0, y)$ funksiya grafigiga $M_0(x_0, y_0, f(x_0, y_0))$ nuqtada o‘tkazilgan urinma orasidagi burchak.

7.5. Funksiyaning differensiallanuvchanligi va to‘la differensiali

$z = f(x,y)$ funksiyaning $M(x,y)$ nuqtadagi to‘la orttirmasi deb
$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$$
miqdorga aytilishini eslatib o‘tamiz.

7.19-Ta’rif. Agar $z = f(x,y)$ funksiyaning $M(x,y)$ nuqtadagi to‘la orttirmasini
$$\Delta z = A \cdot \Delta x + B \cdot \Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y \qquad (7.3)$$

ko‘rinishda ifodalash mumkin bo‘lsa, bu funksiya M nuqtada differensiallanuvchi deyiladi, bu yerda A va B lar Δx va Δy orttirmalarga bog‘liq bo‘lmagan sonlar, $\alpha(\Delta x, \Delta y)$ va $\beta(\Delta x, \Delta y)$ esa $\Delta x \to 0$, $\Delta y \to 0$ da cheksiz kichik, ya’ni

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \alpha(\Delta x, \Delta y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \beta(\Delta x, \Delta y) = 0.$$

7.3-Teorema (Differensiallanuvchanlikning zaruriy sharti). Agar $z = f(x, y)$ funksiyz $M(x, y)$ nuqtada differensiallanuvchi bo‘lsa, u bu nuqtada uzluksiz hamdir.

► Agar $z = f(x, y)$ funksiyz $M(x, y)$ nuqtada differensiallanuvchi bo‘lsa, (7.3) tenglikdan $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = 0$ ekanligi kelib chiqadi. Bu esa 7.17-Ta’rifga muvofiq, $z = f(x, y)$ funksiyaning $M(x, y)$ nuqtada uzluksiz ekanligini angalatadi.◄

7.4-Teorema. Agar $z = f(x, y)$ funksiya $M(x, y)$ nuqtada differensiallanuvchi bo‘lsa, u bu nuqtada $f'_x(x, y)$ va $f'_y(x, y)$ xususiy hosilalarga ega va

$$f'_x(x, y) = A,\ f'_y(x, y) = B \tag{7.4}$$

bo‘ladi.

► $z = f(x, y)$ funksiyz $M(x, y)$ nuqtada differensiallanuvchi bo‘lganligi uchun (7.3) munosabat o‘rinli, bu yerda $\Delta y = 0$ deb olsak, $\Delta_x z = A \cdot \Delta x + \alpha(\Delta x, 0)\Delta x$ tenglikka ega bo‘lamiz. Bu tenglikning ikkala tomonini Δx orttirmaga bo‘lib va $\Delta x \to 0$ da limitga o‘tsak

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} [A + \alpha(\Delta x, 0)] = A$$

tenglikni hosil qilamiz. Demak 7.18-Ta’rifga ko‘ra $f'_x(x, y) = A$ xususiy hosila mavjud bo‘ladi.

Xuddi shu singari $M(x, y)$ nuqtada $f'_y(x, y) = B$ xususiy hosilaning mavjudligi isbotlanadi. ◄

Ikki o‘zgaruvchili funksiya differensiallanuvchanligining yetarli sharti. Ma’lumki $y = f(x)$ bir o‘zgaruvchili funksiyaning x_0 nuqtada differensiallanuvchanligining zaruriy va yetarli sharti bu nuqtada $f'(x)$ chekli hosilaning mavjudligidan iborat. Funksiya ikki o‘zgaruvchiga bog‘liq bo‘lganda bunday shart yo‘q. Yuqorida qaralgan alohida zaruriy shart va alohida yetarli shart mavjud xolos. Ikki o‘zgaruvchili funksiya differensiallanuvchanligining yetarli sharti quyidagi teoremada ifodalangan.

7.5-Torema. Agar $z = f(x,y)$ funksiya $M(x,y)$ nuqtaning biror atrofida f'_x va f'_y xususiy hosilalarga ega va bu hosilalar M nuqtada uzluksiz bo‘lsa, $z = f(x,y)$ funksiya $M(x,y)$ nuqtada differensiallanuvchi bo‘ladi.

► x va y o‘zgaruvchilarga $M_1(x+\Delta x, y+\Delta y)$ nuqta M nuqtaning teorema shartida aytib o‘tilgan atrofidan chiqib ketmaydigan qilib Δx va Δy orttirmalar beramiz. Funksiyaning

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

to‘la orttirmasini

$$\Delta z = [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + \\ = [f(x, y+\Delta y) - f(x,y)] \qquad (7.5)$$

ko‘rinishda yozib olamiz. Birinchi qavs ichidagi $f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)$ ifodani bitta x o‘zgaruvchining $f(x, y+\Delta y)$ funksiyasining orttirmasi sifatida qarash mumkin (ikkinchi argument o‘zgarmas $y+\Delta y$ o‘zgarmas qiymatga ega). Shartga ko‘ra bu funksiya $f'_x(x, y+\Delta y)$ hosilaga ega. U holda 3.12-Lagranj teoremasiga ko‘ra

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = f'_x(x+\theta_1\Delta x, y+\Delta y)\Delta x,$$

bo‘ladi, bu yerda $0 < \theta_1 < 1$. $f(x, y+\Delta y) - f(x,y)$ ifodaga nisbatan xuddi shunday mulohaza yuritib

$$f(x, y+\Delta y) - f(x,y) = f'_y(x, y+\theta_2\Delta y)\Delta y, \qquad (0 < \theta_2 < 1)$$

tenglikni yozamiz. f'_x va f'_y hosilalar $M(x,y)$ nuqtada uzluksiz bo‘lganligi uchun

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_x(x+\theta_1\Delta x, y+\Delta y) = f'_x(x,y),$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_y(x, y+\theta_2\Delta y) = f'_y(x,y)$$

bo‘ladi. Bundan esa

$$f'_x(x+\theta_1\Delta x, y+\Delta y) = f'_x(x,y) + \alpha(\Delta x, \Delta y),$$

$$f'_y(x, y+\theta_2\Delta y) = f'_y(x,y) + \beta(\Delta x, \Delta y)$$

ekanligi kelib chiqadi, bu yerda $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \alpha(\Delta x, \Delta y) = \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \beta(\Delta x, \Delta y) = 0$. Hosil qilingan bu ifodalarni (7.5) tenglikka qoysak, Δz to‘la orttirma uchun

$$\Delta z = f'_x(x,y)\Delta x + f'_y(x,y)\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y$$

tenglikni hosil qilamiz. Bu esa $z = f(x, y)$ funksiyaning $M(x, y)$ nuqtada differensiallanuvchi ekanligini anglatadi.◄

Funksiyaning differensiali. Agar $z = f(x, y)$ funksiya $M(x, y)$ nuqtada differensiallanuvchi bo‘lsa, uning bu nuqtadagi to‘la differensialini

$$\Delta z = A \cdot \Delta x + B \cdot \Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y$$

ko‘rinishda tasvirlash mumkinligini eslatib o‘tamiz, bu yerda $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \alpha(\Delta x, \Delta y) = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \beta(\Delta x, \Delta y) = 0$.

7.20-Ta’rif. $z = f(x, y)$ funksiyaning $M(x, y)$ nuqtadagi dz differensiali deb, uning bu nuqtadagi to‘la orttirmasining Δx va Δy orttirmalarga nisbatan chiziqli qismiga aytiladi, ya’ni

$$dz = A \cdot \Delta x + B \cdot \Delta y. \tag{7.6}$$

7.3-Teoremani qo‘lab (7.6) tenglikni

$$dz = f'_x(x, y)\Delta x + f'_y(x, y)\Delta y \tag{7.7}$$

ko‘rinishda yozish mumkin.

Erkin x va y o‘zgaruvchilarning differensiali deb ularning orttirmalariga aytamiz: $dx = \Delta x$, $dy = \Delta y$. U holda funksiyaning differensiali

$$dz = f'_x(x, y)dx + f'_y(x, y)dy \tag{7.8}$$

ko‘rinishda bo‘ladi.

To‘la differensialning taqribiy hisobga tadbiqi. $z = f(x, y)$ funksiya differensiali ta’rifidan yetarlicha kichik $|\Delta x|$ va $|\Delta y|$ qiymatlarda

$$\Delta z \approx dz \tag{7.9}$$

taqribiy tenglik o‘rinli bo‘ladi. To‘la orttirmaning $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ formulasidan (7.9) taqribiy tenglikni

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx dz$$

ko‘rinishda yozish mumkin. (7.7) tenglikni inobatga olsak, so‘ngi tenglikdan

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f'_x(x, y)\Delta x + f'_y(x, y)\Delta y \tag{7.10}$$

taqribiy hisoblash formulasini hosil qilamiz.

7.12-Misol. $1{,}02^{3{,}01}$ ifodaning qiymatini taqribiy hisoblang.

►$z = x^y$ funksiyani qaraymiz. U holda $1{,}02^{3{,}01} = (x + \Delta x)^{y + \Delta y}$, bu yerda $x = 1$, $\Delta x = 0{,}02$, $y = 3$, $\Delta y = 0{,}01$. Qaralayotgan funksiyaning xususiy hosilalarini topamiz: $z'_x = (x^y)'_x = y \cdot x^{y-1}$, $z'_y = (x^y)'_y = x^y \cdot \ln x$. (7.10) taqribiy tenglikka ko‘ra: $1{,}02^{3{,}01} \approx 1^3 + 3 \cdot 1^{3-1} \cdot 0{,}02 + 1^3 \cdot \ln 1 \cdot 0.01 = 1{,}06$.◄

7.6. Murakkab funksiyaning hosilalari

$z = f(x, y)$ ikki o‘zgaruvchili funksiyaning x va y o‘zgaruvchilari o‘z navbatida erkin t o‘zgaruvchining funksiyalari bo‘lsin: $x = x(t), y = y(t)$. U holda $z = f[x(t), y(t)]$ funksiya erkin t o‘zgaruvchining funksiyasi bo‘ladi.

7.6-Teorema. Agar $x = x(t)$ va $y = y(t)$ funksiyalar t nuqtada differensiallanuvchi, $z = f(x, y)$ funksiya esa $M(x(t), y(t))$ nuqtada differensiallanuvchi bo‘lsa, $z = f[x(t), y(t)]$ murakkab funksiya ham t nuqtada differensiallanuvchi bo‘ladi va uning hosilasi

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad (7.11)$$

formulaga ko‘ra hisoblanadi.

►t o‘zgaruvchiga Δt orttirma beramiz, u holda $x(t)$ va $y(t)$ funksiyalar mos ravishda Δx va Δy orttirmalar olishadi, $z = f(x, y)$ funksiya ham o‘z navbatida

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

orttirma oladi.

$z = f(x, y)$ funksiya $M(x, y)$ nuqtada differensiallanuvchi bo‘lganligi (bu yerda $x = x(t), y = y(t)$) uchun, Δz orttirmani

$$\Delta z = f'_x(x, y)\Delta x + f'_y(x, y)\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y$$

ko‘rinishda yozish mumkin, bu yerda $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \alpha(\Delta x, \Delta y) = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \beta(\Delta x, \Delta y) = 0$. Qo‘shimcha ravishda $\alpha(0,0) = \beta(0,0) = 0$ deb olib, $\alpha(\Delta x, \Delta y)$ va $\beta(\Delta x, \Delta y)$ cheksiz kichiklarni $\Delta x = 0, \Delta y = 0$ da aniqlab olamiz.

Δz uchun hosil qilingan ifodaning ikkala tomonini Δt orttirmaga bo‘lamiz:

$$\frac{\Delta z}{\Delta t} = f'_x(x, y)\frac{\Delta x}{\Delta t} + f'_y(x, y)\frac{\Delta y}{\Delta t} + \alpha(\Delta x, \Delta y)\frac{\Delta x}{\Delta t} + \beta(\Delta x, \Delta y)\frac{\Delta y}{\Delta t}. \qquad (7.12)$$

Shartga ko‘ra $\lim\limits_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \lim\limits_{\Delta t \to 0}\frac{\Delta y}{\Delta t} = \frac{dy}{dt}$. Bundan tashqari $x(t), y(t)$ funksiyalar t nuqtada differensiallanuvchanligidan, ularning bu nuqtada uzluksizligi ham kelib chiqadi, ya’ni $\lim\limits_{\Delta t \to 0}\Delta x = \lim\limits_{\Delta t \to 0}\Delta y = 0$ bo‘ladi. Shuning uchun bu munosabatlardan $\lim\limits_{\Delta t \to 0}\alpha(\Delta x, \Delta y) = \lim\limits_{\Delta t \to 0}\beta(\Delta x, \Delta y) = 0$ tenglikni yozish mumkin. U holda Δt nolga intilganda (7.12) tenglikdagi uchinchi va to‘rtinchi hadlar ham nolga intiladi:

$$\lim_{\Delta t\to 0}\alpha(\Delta x,\Delta y)\frac{\Delta x}{\Delta t}=0,\ \lim_{\Delta t\to 0}\beta(\Delta x,\Delta y)\frac{\Delta y}{\Delta t}=0.$$

Shunday qilib, Δt nolga intilganda (7.12) tenglikning o‘ng tomoni chekli limitga ega va demak chap tomoni ham chekli limitga ega bo‘ladi:

$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\frac{dz}{dt}$$

va shu bilan birga

$$\frac{dz}{dt}=f'_x(x,y)\frac{dx}{dt}+f'_y(x,y)\frac{dy}{dt}\ \text{ yoki }\ \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

7.13-Misol. $z=x^3y^4$, $x=2t$, $y=t^2$ bo‘lsa, $\frac{dz}{dt}$ hosilani toping.

► $z'_x(x,y)=3x^2y^4$, $z'_y(x,y)=4x^3y^3$, $x'(t)=2$, $y'(t)=2t$ hosilalarni (7.11) formulaga qo‘yamiz:

$$\frac{dz}{dt}=3x^2y^4\cdot 2+4x^3y^3\cdot 2t.$$

$x=2t$, $y=t^2$ ekanligini inobatga olsak,

$$\frac{dz}{dt}=3(2t)^2(t^2)^4 2+4(2t)^3(t^2)^3 2t=88t^{10}.$$

Boshqa tomondan, dastlab z ni t orqali ifodalab olib, so‘ngra $\frac{dz}{dt}$ hosilani olish mumkin: $z=(2t)^3(t^2)^4=8t^{11}, \frac{dz}{dt}=88t^{10}$, ya’ni (7.11) formula bo‘yicha olingan natija bilan bir xil ekan.◄

Agar $z=f(x,y)$ funksiyada $y=\varphi(x)$ bo‘lsa, $z=f[x,\varphi(x)]$ funksiya x o‘zgaruvchiga bog‘liq bo‘ladi. Bu yerda t parametr o‘rnida x o‘garuvchi bo‘ladi, shuning uchun (7.11) formulaga ko‘ra

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dx}+\frac{\partial z}{\partial y}\frac{dy}{dx}$$

va bu yerda $\frac{dx}{dx}=1$ bo‘lganligi uchun, so‘ngi tenglik

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}$$

ko‘rinishni oladi.

Endi umumiyroq holni qarab chiqamiz. $z=f(x,y)$ funksiyaning x va y o‘zgaruvchilari o‘z navbatida ikkita yoki undan ko‘proq erkin o‘zgaruvchiga bog‘liq bo‘lsin. Masalan, $x=x(u,v)$, $y=y(u,v)$ bo‘lsin. U holda $z=$

$f[x(u,v),y(u,v)]$ erkin u va v o‘zgaruvchilarning murakkab funksiyasi bo‘ladi.

Agar $x(u,v)$ va $y(u,v)$ funksiyalar $M'(u,v)$ nuqtada differensiallanuvchi, $z=f(x,y)$ funksiya esa $x=x(u,v)$, $y=y(u,v)$ bo‘lgan $M(x,y)$ nuqtada differensiallanuvchi bo‘lsa, murakkab $z=f[x(u,v),y(u,v)]$ funksiya ham $M'(u,v)$ nuqtada differensiallanuvchi bo‘ladi va uning xususiy hosilalari

$$\begin{cases} \dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} \end{cases} \tag{7.13}$$

formulalarga ko‘ra topiladi.

7.14-Misol. $z=x^2y^2, x=u+v, y=\dfrac{u}{v}$ bo‘lsa $\dfrac{\partial z}{\partial u},\dfrac{\partial z}{\partial v}$ xususiy hosilalar topilsin.

► $z'_x=2xy^2,\ z'_y=2x^2y,\ x'_u=1, x'_v=1,\ y'_u=\dfrac{1}{v},\ y'_v=-\dfrac{u}{v^2}$ xususiy hosilalarni (7.12) formulalarga qo‘yamiz:

$$\frac{\partial z}{\partial u}=2xy^2\cdot 1+2x^2y\cdot\frac{1}{v}, \qquad \frac{\partial z}{\partial v}=2xy^2\cdot 1+2x^2y\cdot\left(-\frac{u}{v^2}\right).$$

Bu ifodalarda x va y o‘zgaruvchilarni u va v orqali ifodalaymiz:

$$\frac{\partial z}{\partial u}=2(u+v)\frac{u^2}{v^2}+2(u+v)^2\frac{u}{v}\cdot\frac{1}{v}=2(u+v)\frac{2u^2+uv}{v^2};$$

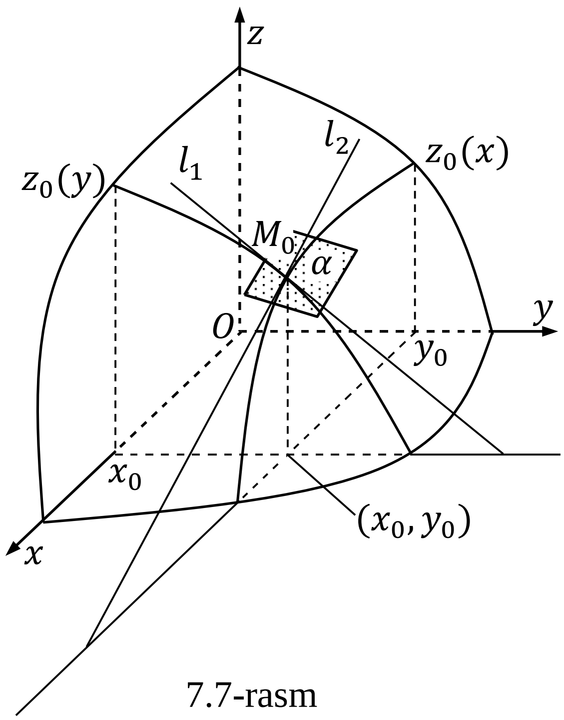
$$\frac{\partial z}{\partial v}=2(u+v)\frac{u^2}{v^2}-2(u+v)^2\frac{u}{v}\cdot\frac{u}{v^2}=-\frac{2(u+v)u^3}{v^3}.$$

$z=x^2y^2$ funksiyada dastlab x va y o‘zgaruvchilarni u va v erkin o‘zgaruvchilar orqali ifodalab, so‘ngra hosil bo‘lgan funksiyadan u va v bo‘yicha xususiy hosilalar olsak, xuddi shu natijaga kelishni tekshirib ko‘rishni kitobxonga havola qilamiz.◄

7.7. Sirtga o‘tkazilgan urinma tekislik va normal

Ikki o‘zgaruvchili funksiya xususiy hosilalarining geometrik tadbiqlaridan birini qaraymiz. Biror $D\subset\mathbb{R}^2$ sohaga tegishli bo‘lgan (x_0,y_0) nuqtada $z=f(x,y)$ funksiya differensiallanuvchi bo‘lsin. Bu funksiyani tasvirlovchi σ sirtni $x=x_0$ va $y=y_0$ tekisliklar bilan kesamiz (7.7-

rasm). $x = x_0$ tekislik σ sirtni biror $z_0(y)$ chiziq bo‘ylab kessin. Uning tenglamasi berilgan $z = f(x, y)$ funksiya ifodasida x o‘rniga x_0 qo‘yish bilan hosil qilinadi.



7.7-rasm

$M_0(x_0, y_0, f(x_0, y_0))$ nuqta bu $z_0(y)$ chiziqda yotadi. z funksiyaning M_0 nuqtada differensiallanuvchanligidan $z_0(y)$ funksiyaning ham $y = y_0$ nuqtada differensiallanuvchanligi kelib chiqadi. Shuning uchun bu nuqtada $x = x_0$ tekislikda $z_0(y)$ chiziqqa l_1 urima o‘tkazish mumkin.

$y = y_0$ kesim uchun ham xuddi shu singari mulohaza yuritib, $x = x_0$ nuqtada $z_0(x)$ chiziqqa l_2 urima o‘tkazamiz. l_1 va l_2 to‘g‘ri chiziqlar α tekislikni aniqlaydi. Bu tekislik σ sirtga M_0 nuqtada o‘tkazilgan *urinma tekislik* deb ataladi.

Bu tekislik tenglamasini tuzamiz. α tekislik $M_0(x_0, y_0, z_0)$ nuqta orqali o‘tganligi uchun, uning tengalamasini

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

yoki

$$z - z_0 = A_1(x - x_0) + B_1(y - y_0) \tag{7.14}$$

ko‘rinishda yozish mumkin. So‘ngi tenglamadagi A_1 va B_1 koffisiyentlarni topamiz.

l_1 va l_2 urinmalar mos ravishda

$$z - z_0 = f'_y(x_0, y_0) \cdot (y - y_0), \quad x = x_0;$$
$$z - z_0 = f'_x(x_0, y_0) \cdot (x - x_0), \quad y = y_0$$

tenglamalarga ega bo‘lishadi.

l_1 urinma α tekislikda yotadi, shuning uchun l_1 to‘g‘ri chiziq nuqtalarining koordinatalari (7.14) tenglamani qanoatlantiradi, ya’ni bu nuqtalarda

$$\begin{cases} z - z_0 = f'_y(x_0, y_0) \cdot (y - y_0), \\ x = x_0, \\ z - z_0 = A_1(x - x_0) + B_1(y - y_0) \end{cases}$$

sistema o‘rinli bo‘ladi. Bu sistemani B_1 noma’lumga nisbatan yechib, $B_1 = f'_y(x_0, y_0)$ tenglikni hosil qilamiz.

l_2 urinmaga nisbatan ham xuddi shunday mulohaza yuritib $A_1 = f'_x(x_0, y_0)$ tenglikka ega bo‘lamiz.

A_1 va B_1 noma’lumlarning topilgan qiymatlarini (7.14) formulaga qo‘yib, qidirilayotgan urinma tekislikning

$$z - z_0 = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0) \qquad (7.15)$$

tenglamasini hosil qilamiz.

M_0 nuqtadan urinma tekislikka perpendikulyar bo‘lib o‘tuvchi to‘g‘ri chiziq sirtning *normali* deb ataladi.

To‘g‘ri chiziq va tekislikning perpendikulyarlik shartidan foydalanib (2.13 ga qarang), normalning

$$\frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)} = \frac{z - z_0}{-1} \qquad (7.16)$$

kanonik tenglamasini hosil qilamiz.

Mulohaza. Sirtning urinma tekisligi va normalining tenglamalari xususiy hosilalar noldan farqli bo‘lgan maxsusmas nuqtalar uchun keltirb chiqarildi. Xususiy hosilalar nolga aylanadigan *maxsus nuqtalarni* qaramaymiz.

7.15-Misol. $z = xy$ sirtga $M_0(-1,2,-2)$ nuqtada o‘tkzilgan urinma tekislik va normal to‘g‘ri chiziq tenglamasini tuzing.

►Xususiy hosilalrni topamiz: $f'_x(x, y) = y$, $f'_y(x, y) = x$ va ularning $(-1,2)$ nuqtadagi qiymatlari $f'_x(-1,2) = 2$, $f'_y(-1,2) = -1$ bo‘ladi. U holda (7.15) va (7.16) formulalarga ko‘ra urinma tekislik va normal mos ravishda

$$z + 2 = 2(x + 1) - (y - 2) \text{ yoki } 2x - y - z - 2 = 0 \text{ va}$$

$$\frac{x + 1}{2} = \frac{y - 2}{-1} = \frac{z + 2}{-1}$$

tenglamalarga ega bo‘ladi.◄

7.8. Yuqori tartibli xususiy hosilalar va differensiallar

Yuqori tartibli xususiy hosilalar. $M(x, y)$ nuqtaning biror atrofida aniqlangan $z = f(x, y)$ funksiyaning $f'_x(x, y)$ va $f'_y(x, y)$ xususiy hosilalari bu atrofning barcha nuqtalarida mavjud bo‘lsin. Bu holda xususiy hosilalar x va y o‘zgaruvchilarning M nuqta atrofida aniqlangan funksiyalaridan iborat bo‘ladi. Ularni biz *birinchi tartibli xususiy hosilalar* deb ataymiz.

O‘z navbatida M nuqtada $f'_x(x,y)$ va $f'_y(x,y)$ funksiyalardan x va y o‘zgaruvchilar bo‘yicha xususiy hosilalar mavjud bo‘lsa, ularni biz $f(x,y)$ funksiyaning bu nuqtadagi ikkinchi tartibli xususiy hosilalari deb ataymiz va

$$\frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y); \quad \frac{\partial^2 z}{\partial y \partial x} = f''_{yx}(x,y);$$
$$\frac{\partial^2 z}{\partial y^2} = f''_{yy}(x,y); \quad \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y)$$

orqali belgilaymiz.

$f''_{yx}(x,y), f''_{xy}(x,y)$ ko‘rinishdagi ikkinchi tartibli xususiy hosilalar *aralash xususiy hosilalar* deb ataladi.

7.16-Misol. $z = x^2y^3 + 4x - 5y$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

►Dastlab birinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x} = 2xy^3 + 4, \qquad \frac{\partial z}{\partial y} = 3x^2y^2 - 5.$$

Hosil qilingan funksiyalardan yana x va y bo‘yicha xususiy hosilalar olib:

$$\frac{\partial^2 z}{\partial x^2} = 2y^3, \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 6xy^2, \qquad \frac{\partial^2 z}{\partial y^2} = 6x^2y.$$

ikkinchi tartibli xususiy hosilalarni hosil qilamiz.◄

7.7-Teorema. Agar $M(x,y)$ nuqtaning δ atrofida $f''_{yx}(x,y)$ va $f''_{xy}(x,y)$ aralash xususiy hosilalar mavjud va M nuqtaning o‘zida uzluksiz bo‘lsa, ular bu nuqtada o‘zaro teng bo‘ladi, ya’ni

$$f''_{xy}(x,y) = f''_{yx}(x,y) \tag{7.17}$$

tenglik o‘rinli bo‘ladi.

► $A = [f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)] - [f(x, y+\Delta y) - f(x,y)]$ ifodani qaraymiz, bu yerda Δx va Δy lar $M_1(x+\Delta x, y+\Delta y)$ nuqta teorema shartida aytib o‘tilgan M nuqtaning δ atrofiga bo‘ladigan qilib olingan yetarlicha kichik sonlar.

$$\varphi(x) = f(x, y+\Delta y) - f(x,y)$$

yordamchi funksiyani kiritamiz. U holda A ifodani $[x, x+\Delta x]$ kesmada differensiallanuvchi bo‘lgan $\varphi(x)$ bir o‘zgaruvchili funksiyaning orttirmasi sifatida qarash mumkin:

$$A = \Delta\varphi = \varphi(x+\Delta x) - \varphi(x).$$

Shuning uchun bu ayirmaga Lagranj teoremasini qo‘llab

$$A=\Delta\varphi=\varphi'(x+\theta_1\Delta x)\Delta x=$$
$$=[f'_x(x+\theta_1\Delta x,y+\Delta y)-f'_x(x+\theta_1\Delta x,y)]\Delta x,\ \ 0<\theta_1<1$$
tenglikni hosil qilamiz. Kvadrat qavsning ichidagi ifodani $[y,y+\Delta y]$ kesmada differensiallanuvchi bo‘lgan bitta y o‘zgaruvchili $f'_x(x+\theta_1\Delta x,y)$ funksiyaning orttirmasi sifatida qarash mumkin. Yana Lagranj teoremasini (y o‘zgaruvchi bo‘yicha) qo‘llasak
$$A=f''_{yx}(x+\theta_1\Delta x,y+\theta_2\Delta y)\Delta x\Delta y,\ 0<\theta_1,\theta_2<1 \qquad (7.18)$$
tenglik hosil bo‘ladi.

Boshqa tomondan, agar
$$\psi(y)=f(x,y+\Delta y)-f(x,y)$$
yordamchi funksiyani kiritsak, uning uchun
$$A=\Delta\psi=\psi(y+\Delta y)-\psi(y)$$
tenglik o‘rinli bo‘lishini osongina ko‘rsatish mumkin. So‘ngra $\varphi(x)$ funksiyaga nisbatan qilingan mulohazalarni takrorlab
$$A=f''_{xy}(x+\theta_4\Delta x,y+\theta_3\Delta y)\Delta x\Delta y,\ 0<\theta_3,\theta_4<1 \qquad (7.19)$$
tenglikni hosil qilamiz. (7.18) va (7.19) tengliklarni taqqoslab
$$f''_{yx}(x+\theta_1\Delta x,y+\theta_2\Delta y)=f''_{xy}(x+\theta_4\Delta x,y+\theta_3\Delta y)$$
tenglikka ega bo‘lamiz. $f''_{yx}(x,y)$ va $f''_{xy}(x,y)$ xususiy hosilalarning M nuqtada uzluksizligini inobatga olib, so‘ngi tenglikda Δx va Δy orttirmalarni nolga intiltirsak
$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f''_{yx}(x+\theta_1\Delta x,y+\theta_2\Delta y)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f''_{xy}(x+\theta_4\Delta x,y+\theta_3\Delta y)$$
yoki
$$f''_{xy}(x,y)=f''_{yx}(x,y)$$
tenglik hosil bo‘ladi.◄

Ikkinchi tartibli xususiy hosilalardan yana x va y bo‘yicha xususiy hosilalar olib, uchinchi tartibli va bu jarayonni davom ettirib $n-$taribli xususiy hosilalar kiritiladi va aralash hosilalarning tengligi haqidagi 7.5-Teoremaga o‘xshash teoremani isbotlash mumkin bo‘ladi.

Yuqori tartibli differensiallar. 7.5. bo‘olimda differensiallanuvchi funksiyaning differensiali tushunchasini kiritgan va
$$dz=f'_x(x,y)dx+f'_y(x,y)dy \qquad (7.20)$$
formulani hosil qilgan edik. Uni biz *birinchi tartibli differensial* deb ataymiz.

$f'_x(x,y)$ va $f'_y(x,y)$ funksiyalar $M(x,y)$ nuqtada differensiallanuvchi bo‘lsin. dz uchun hosil qilingan yuqoridagi ifodada dx va dy differensiallarni o‘zgarmaslar sifatida qaraymiz. U holda dz differensial faqat x va y o‘zgaruvchilarning funksiyasidan iborat bo‘ladi va unga nisbatan (7.20) formulani qo‘llash mumkin bo‘ladi, natijada d^2z *ikkinchi tartibli differensialni* hosil qilamiz:

$$d^2z = d(dz) = d\big(f'_x(x,y)dx + f'_y(x,y)dy\big) =$$

$$= \left[f'_x(x,y)dx + f'_y(x,y)dy\right]'_x dx + \left[f'_x(x,y)dx + f'_y(x,y)dy\right]'_y dy =$$

$$= f''_{xx}(x,y)(dx)^2 + f''_{xy}(x,y)dxdy + f''_{yx}(x,y)dydx + f''_{yy}(x,y)(dy)^2$$

Agar f''_{xy} va f''_{yx} aralash hosialar uzluksiz bo‘lsa, 7.5-Teoremaga ko‘ra

$$f''_{xy}(x,y) = f''_{yx}(x,y)$$

bo‘ladi. U holda d^2z ikkinchi tartibli differensial uchun hosil qilingan ifoda

$$d^2z = f''_{xx}(dx)^2 + 2f''_{xy}dxdy + f''_{yy}(dy)^2 \qquad (7.21)$$

ko‘rinishni oladi.

Xuddi shu singari

$$d^3z = f'''_{x^3}(dx)^3 + 3f'''_{x^2y}(dx)^2dy + 3f'''_{xy^2}dx(dy)^3 + f'''_{y^3}(dy)^3$$

....

$$d^nz = f^{(n)}_{x^n}(dx)^n + nf^{(n)}_{x^{n-1}y}(dx)^{n-1}dy + \cdots$$

$$\ldots + \frac{n(n-1)\ldots(n-k+1)}{k!}f^{(n)}_{x^{n-k}y^k}(dx)^{n-k}(dy)^k + \cdots + f^{(n)}_{y^n}(dy)^n$$

yuqori tartibli differensiallarni hosil qilamiz.

n tartibli d^nz differesialning formulasi ikkihad n −darajasining Nyuton formulasi bo‘yicha yoyilmasini eslatadi. Shuning uchun d^nz differesialning ifodasini simvolik ravishda

$$d^nz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n f(x,y)$$

ko‘rinishda yozish mumkin.

7.17-Misol. $z = \cos x \sin y$ funksiyaning uchinchi tartibli d^3z differensialini toping.

►Uchinchi tartibli xususiy hosilalarni topamiz:

$$z'_x = -\sin x \sin y\,, z'_y = \cos x \cos y,$$

$$z''_{x^2} = -\cos x \sin y\,, z''_{y^2} = -\cos x \sin y,$$

$$z''_{xy} = -\sin x \cos y\,, z'''_{x^3} = \sin x \sin y,$$

$$z'''_{y^3} = -\cos x \cos y\,, z'''_{x^2y} = -\cos x \cos y\,, z'''_{y^2x} = \sin x \sin y.$$

Ularni uchinchi tartibli differensial formulasiga qo‘yamiz:

$$d^3z = \sin x \sin y\,(dx)^3 - 3\cos x \cos y\,(dx)^2 dy +$$
$$+3\sin x \sin y\, dx(dy)^2 - \cos x \cos y\,(dy)^3. \blacktriangleleft$$

7.9. Ikki o‘zgaruvchili funksiya uchun Teylor formulasi

Bir o‘zgaruvchili funksiya singari ikki o‘zgaruvchili funksiyani ham $n-$darajali ko‘phad bilan birorta qoldiq gadning yig‘indisi ko‘rinishda tasvirlash mumkin.

$z = f(x,y)$ funksiya D sohada $(n+1)-$tartibgacha va bu tartibdagi barcha uzluksiz xususiy hosilalarga ega bo‘lsin. Bu sohada ixtiyoriy $M_0(x_0, y_0) \in D$ nuqtani olamiz va yangi $M(x_0 + \Delta x, y_0 + \Delta y)$ nuqtani qaraymiz. Bunda M_0 va M nuqtalarni tutashtiruvchi kesma butunlay D sohada yotadi deb faraz qilamiz.

$F(t) = f(x_0 + t \cdot \Delta x, y_0 + t \cdot \Delta y)$ tenglik orqali $[0,1]$ kesmada aniqlangan yordamchi funksiyani kiritamiz. Qilingan farazga ko‘ra bu funksiya $[0,1]$ kesmada birinchi, ikkinchi,…, $(n+1)-$ taribli uzluksiz hosilalarga ega. U holda bu funksiya uchun Makloren shaklidagi (5.89) Teylor formulasi o‘rinli:

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \cdots + \frac{F^{(n)}(0)}{n!}t^n +$$
$$+\frac{F^{(n+1)}(\theta t)}{(n+1)!}t^{n+1}, \tag{7.22}$$

bu yerda $\theta \in (0,1)$. Bu formulada $t = 1$ deb osak

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(\theta)}{(n+1)!}, \tag{7.23}$$

bo‘ladi. Boshqa tomondan esa

$$F(1) = f(x_0 + \Delta x, y_0 + \Delta y) \text{ va } F(0) = f(x_0, y_0).$$

(7.11) formulaga ko‘ra

$$F'(t) = f'_x(x_0 + t \cdot \Delta x, y_0 + t \cdot \Delta y)\Delta x + f'_y(x_0 + t \cdot \Delta x, y_0 + t \cdot \Delta y)\Delta y,$$

bundan esa

$$F'(0) = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y = \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right) f(x_0, y_0)$$

tenglikni hosil qilamiz. Ikkinchi tartibli hosila esa

$$F''(t) = f''_{xx}(x_0 + t \cdot \Delta x, y_0 + t \cdot \Delta y)(\Delta x)^2 + 2f''_{xy}(x_0 + t \cdot \Delta x, y_0 +$$
$$+t \cdot \Delta y)\Delta x \Delta y + f''_{yy}(x_0 + t \cdot \Delta x, y_0 + t \cdot \Delta y)(\Delta y)^2$$

ko‘rinishda bo‘ladi va bu tenglikdan

$$F''(0) = f''_{xx}(x_0, y_0)(\Delta x)^2 + 2f''_{xy}(x_0, y_0)\Delta x\Delta y + f''_{yy}(x_0, y_0)(\Delta y)^2 =$$
$$= \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^2 f(x_0, y_0)$$
ikkinchi tartibli hosila uchun ifodaga ega bo‘lamiz. Bu jarayonni davom ettirib,
$$F^{(n)}(0) = \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^n f(x_0, y_0),$$
$$F^{(n+1)}(\theta) = \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^{n+1} f(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$$
formulalarni hosil qilamiz.

Bu ifodalarni (7.22) formulaga qo‘ysak
$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right) f(x_0, y_0) +$$
$$+\frac{1}{2!}\left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^2 f(x_0, y_0) + \cdots + \frac{1}{m!}\left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^n f(x_0, y_0) +$$
$$+\left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^{n+1} f(x_0 + \theta\Delta x, y_0 + \theta\Delta y), 0 < \theta < 1 \qquad (7.24)$$
ikki o‘zgaruvchili funksiya uchun Teylor formulasini hosil qilamiz. Bu formuladagi
$$R_n = \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^{n+1} f(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$$
ifoda *Lagranj shaklidagi qoldiq had* deb ataladi.

Agar $x_0 = y_0 = 0$ bo‘lsa va $\Delta x = x,\ \Delta y = y$ deb olsak (7.24) formula
$$f(x, y) = f(0,0) + \left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) f(0,0) +$$
$$+\frac{1}{2!}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)^2 f(0,0) + \cdots + \frac{1}{m!}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)^n f(0,0) +$$
$$+\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)^{n+1} f(\theta x, \theta y), \qquad 0 < \theta < 1 \qquad (7.25)$$
ko‘rinishni oladi. (7.25) formula *Lagranj qoldiq hadli Makloren formulasi* deb ataladi.

7.18-Misol. $z = e^x \sin y$ funksiyani $n = 2$ hol uchun Makloren formulasi bo‘yicha yoying.

► $n = 2$ hol uchun Makloren formulasi

$$f(x,y)=f(0,0)+f'_x(0,0)x+f'_y(0,0)y+\frac{1}{2!}[f''_{xx}(0,0)x^2+2f''_{xy}(0,0)xy+$$
$$+f''_{yy}(0,0)y^2\big]+\frac{1}{3!}\big[f'''_{xxx}(\theta x,\theta y)x^3+3f'''_{xxy}(\theta x,\theta y)x^2y+$$
$$+3f'''_{xyy}(\theta x,\theta y)xy^2+f'''_{yyy}(\theta x,\theta y)y^3\big],\quad 0<\theta<1 \qquad (7.26)$$

ko‘rinishda bo‘ladi. (7.26) formulaga kirgan koffisiyentlarni hisoblaymiz:

$$\begin{aligned}
f(x,y)&=e^x\sin y & f(0,0)&=0\\
f'_x(x,y)&=e^x\sin y & f'_x(0,0)&=0\\
f'_y(x,y)&=e^x\cos y & f'_y(0,0)&=1\\
f''_{xx}(x,y)&=e^x\sin y & f''_{xx}(0,0)&=0\\
f''_{xy}(x,y)&=e^x\cos y & f''_{xy}(0,0)&=1\\
f''_{yy}(x,y)&=-e^x\sin y & f''_{yy}(0,0)&=0\\
f'''_{xxx}(x,y)&=e^x\sin y & f'''_{xxx}(\theta x,\theta y)&=e^{\theta x}\sin\theta y\\
f'''_{xxy}(x,y)&=e^x\cos y & f'''_{xxy}(\theta x,\theta y)&=e^{\theta x}\cos\theta y\\
f'''_{xyy}(x,y)&=-e^x\sin y & f'''_{xyy}(\theta x,\theta y)&=-e^{\theta x}\sin\theta y\\
f'''_{yyy}(x,y)&=-e^x\cos y & f'''_{yyy}(\theta x,\theta y)&=-e^{\theta x}\cos\theta y.
\end{aligned}$$

Topilganlarni (7.26) formulaga qo‘yib,

$$e^x\sin y=y+xy+\frac{1}{6}\big[e^{\theta x}\sin\theta y\cdot x^3+3e^{\theta x}\cos\theta y\cdot x^2y-$$
$$-3e^{\theta x}\sin\theta y\cdot xy^2-e^{\theta x}\cos\theta y\cdot y^3\big]$$

yoyilmani hosil qilamiz.◄

7.10. Ikki o‘zgaruvchili funksiyaning ekstremumi

Ikki o‘zgaruvchili funksiyaning ekstremumi tushunchasi. $z=f(x,y)$ funksiya biror D sohada aniqlangan va $M_0(x_0,y_0)$ bu sohaning ichki nuqtasi bo‘lsin.

7.21-Ta’rif. Agar shunday $\delta>0$ son topilib, $|\Delta x|<\delta$ va $|\Delta y|<\delta$ tengsizlikni qanoatlantiruvchi barcha Δx va Δy uchun

$$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\le 0 \qquad (7.27)$$

tengsizlik o‘rinli bo‘lsa, $M_0(x_0,y_0)$ nuqta $f(x,y)$ funksiyaning *lokal maksimum nuqtasi* deb ataladi; agar $|\Delta x|<\delta$ va $|\Delta y|<\delta$ tengsizlikni qanoatlantiruvchi barcha Δx va Δy uchun

$$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\ge 0 \qquad (7.28)$$

tengsizlik o‘rinli bo‘lsa, $M_0(x_0,y_0)$ nuqta *lokal minimum nuqta* deb ataladi.

Boshqacha qilib aytganda, $M_0(x_0, y_0)$ nuqtaning shunday δ atrofi topilib, bu atrofdan olingan barcha $M(x, y)$ nuqtalarda funksiyaning

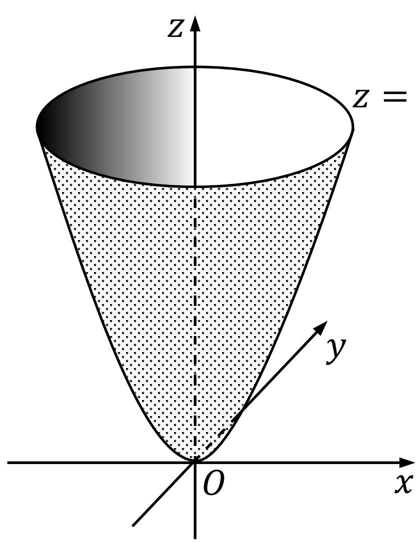
$$\Delta f = f(x, y) - f(x_0, y_0)$$

orttirmasi ishorasini o‘zgartirmasa, $M_0(x_0, y_0)$ maksimum yoki minimum nuqta bo‘ladi.

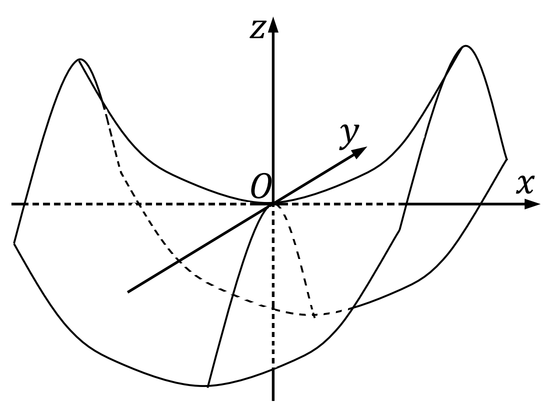
7.19-Misol. $z = x^2 + y^2$ funksiya uchun $O(0,0) -$ minimum nuqta bo‘ladi (7.8-rasm).

Biz $f(x, y)$ funksiya uchun $\Delta f > 0$ yoki $\Delta f < 0$ qat’iy tengsizlik bajariladigan funksiyaning qat’iy maksimum yoki qat’iy minimum nuqtalarnigina qaraymiz.

Funksiyaning maksimum nuqtadagi qiymatini funksiyaning *maksimumi*, minimum nuqtadagi qiymatini esa uning *minimumi* deb ataymiz. Maksimum va minimum nuqtalar funksiyaning *ekstremum nuqtalari*, funksiyaning maksimum va minimumlari esa uning *ekstremumlari* deb ataladi.



7.8-rasm



7.9-rasm

7.8-Teorema (ekstremumning zaruriy sharti). Agar $z = f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada ekstremumga erishsa, bu nuqtada uning z'_x va z'_y xususiy hosilalari yoki nolga aylanadi, yoki mavjud emas.

► y o‘zgaruvchiga $y = y_0$ qiymatni beramiz. U holda $f(x, y_0)$ bitta x o‘zgaruvchining funksiyasiga aylanadi. Shartga ko‘ra u $x = x_0$ nuqtada ekstremumga (maksimumga yoki minimumga) erishadi, demak bir o‘zgaruvchili funksiya ekstremumining zaruriy sharti haqidagi 6.4 va 6,5-Teoremalarga ko‘ra $z'_x(x_0, y_0)$ nolga teng yoki mavjud emas. Xuddi shu singari $z'_y(x_0, y_0)$ nolga tengligi yoki mavjud emasligi isbotlanadi.◄

Bu teorema funksiyaning ekstremal qiymatini o‘rganish uchun yetarli emas, biroq maksimum yoki minimumning mavjudligi oldindan ma’lum bo‘lsa, bunday nuqtalarni topish imkonini beradi. Aks holda boshqa tadqiqotlarni olib borishni talab qilinadi.

Masalan, $z = x^2 - y^2$ funksiya $O(0,0)$ nuqtada nolga aylanadigan $z'_x = 2x$ va $z'_y = -2y$ xususiy hosilalarga ega. Biroq bu funksiya koordinatalar boshida maksimumga ham, minimumga ham erishmaydi. Haqiqatdan ham, u koordinatalar boshida nol qiymatni qabul qiladi va koordinatalar boshining ixtiyoriy kichik atrofida ham manfiy, ham musbat qiymatlarni qabul qiladi. Demak nol qiymat maksimum qiymat ham, minimum qiymat ham bo‘lmaydi (7.9-rasm).

Xususiy hosilalar nolga aylanadigan, ya’ni $z'_x = 0$, $z'_y = 0$ bo‘ladigan yoki ular mavjud bo‘lmaydigan nuqtalarni $z = f(x, y)$ funksiyaning *kritik nuqtalari* deb ataymiz. Demak 7.7-Teoremaga ko‘ra, agar funksiya biror nuqtada ekstremumga erishadigan bo‘lsa, bu nuqta faqat kritik nuqta bo‘lishi mumkin.

7.9-Teorema (ekstremumning yetarli sharti). $z = f(x, y)$ funksiya $M_0(x_0, y_0)$ kritik nuqtada va uning biror atrofida uzluksiz birinchi va ikkinchi tartibli xususiy hosilalarga ega bo‘lsin va

$$\Delta = \begin{vmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{xy}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{vmatrix}$$

deb olamiz. U holda:

i) agar $\Delta > 0$ bo‘lsa, funksiya M_0 nuqtada ekstremumga erishadi va shu bilan birga $f''_{xx}(x_0, y_0) < 0$ bo‘lganda maksimumga, $f''_{xx}(x_0, y_0) > 0$ bo‘lganda esa minimumga erishadi;

ii) agar $\Delta < 0$ bo‘lsa, funksiya M_0 nuqtada ekstremumga erishmayadi;

iii) agar $\Delta = 0$ bo‘lsa, funksiya uchun M_0 ekstremum nuqta bo‘lishi ham bo‘lmasligi ham mumkin.

► $f''_{xx}(x_0, y_0) = A$, $f''_{xy}(x_0, y_0) = B$, $f''_{yy}(x_0, y_0) = C$ deb belgilash kiritamiz. U holda $\Delta = AC - B^2$ bo‘ladi. Shartga ko‘ra $f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0$.

i) $\Delta > 0$ va $A > 0$ (yoki $A < 0$) bo‘lsin. U holda $f(x, y)$ funksiyaning $M_0(x_0, y_0)$ nuqtadagi to‘la orttirmasini $n = 1$ hol uchun Teylor formulasi orqali ifodalasak

$$\Delta f = [A'(\Delta x)^2 + 2B'\Delta x\Delta y + C'(\Delta y)^2] \qquad (7.29)$$

tenglikni hosil qilamiz, bu yerda $A' = f''_{xx}(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$, $B' = f''_{xy}(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$, $C' = f''_{yy}(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$, $0 < \theta < 1$. Ikkinchi tartibli xususiy hosilalarning $M_0(x_0, y_0)$ nuqtada uzluksizligi tufayli

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} A' = f''_{xx}(x_0, y_0) = A > 0, \text{ (yoki } A < 0)$$

hamda

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(A'C' - {B'}^2\right) = f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) - \left[f''_{xy}(x_0, y_0)\right]^2 = \Delta > 0$$

bo‘ladi. Shuning uchun yetarlicha kichik Δx va Δy orttirmalarda $A' > 0$ (yoki $A' < 0$) va $A'C' - {B'}^2 > 0$ tengsizliklar o‘rinli bo‘ladi. $A' \neq 0$ bo‘lganligidan (7.29) tenglikni

$$\Delta f = \frac{1}{A'}\left[(A'\Delta x + B'\Delta y)^2 + (A'C' - {B'}^2)(\Delta y)^2\right] \qquad (7.30)$$

ko‘rinishda yozish mumkin. Kvadrat qavs ichidagi ifodaning ishorasi musbat, shuning uchun agar $A' > 0$ ($f''_{xx}(x_0, y_0) > 0$) bo‘lsa, $\Delta f > 0$ va demak $M_0(x_0, y_0)$ nuqta minimum; $A' < 0$ ($f''_{xx}(x_0, y_0) < 0$) bo‘lsa, $\Delta f < 0$ va demak $M_0(x_0, y_0)$ nuqta maksimum nuqta bo‘ladi. i) tasdiq isbot bo‘ladi.

ii) $\Delta < 0$ bo‘lsin.

Agar $A \neq 0$ bo‘lsa, (7.30) tenglik o‘rinli bo‘ladi. Bu holda Δf to‘la orttirma turli ishorali qiymatlarni qabul qilishini ko‘rsatamiz. (7.30) tenglikda $\Delta y = 0$ bo‘lsa,

$$\Delta f = \frac{1}{A'}(A'\Delta x)^2$$

bo‘ladi, ya’ni Δf to‘la orttirma A ning ishorasi bilan bir xil bo‘ladi.

(7.30) tenglikda $A'\Delta x + B'\Delta y = 0$ bo‘lsa

$$\Delta f = \frac{1}{A'}(A'C' - {B'}^2)(\Delta y)^2$$

bo‘ladi, ya’ni Δf to‘la orttirma A ning ishorasi bilan qarama-qarshi bo‘ladi.

Shunday qilib, $M_0(x_0, y_0)$ nuqtaning atrofida Δx va Δy ning turli qiymatlarida Δf turli ishorali qiymatlarni qabul qilar ekan. Bu esa $M_0(x_0, y_0)$ nuqtaning ekstremal nuqta emasligidan dalolat beradi.

$A = 0, C \neq 0$ bo‘lsin, u holda (7.29) tenglik

$$\Delta f = [2B'\Delta x\Delta y + C'(\Delta y)^2]$$

ko‘rinishda bo‘ladi. Uni

$$\Delta f = \frac{1}{C'}\left[(B'\Delta x + C'\Delta y)^2 - {B'}^2(\Delta x)^2\right]$$

ko‘rinishda yozamiz. Bu yerda ham $\Delta x = 0$ bo‘lsa, Δf to‘la orttirmaning ishorasi C ning ishorasi bilan bir xil, $B'\Delta x + C'\Delta y = 0$ bo‘lsa, C ning ishorasiga qarama qarshi bo‘ladi.

Nihoyat $A = 0, C = 0$ bo‘lsa

$$\Delta f = B'\Delta x\Delta y$$

bo‘ladi va Δx va Δy orttirmalarning turli qiymatlarida Δf to‘la orttirma har xil ishorali qiymatlarni qabul qiladi.

iii) $\Delta = 0$ hol uchun ikki xil natijali misol keltiramiz.

$z = x^3 + y^3$ funksiya va uning $z'_x = 3x^2$, $z'_y = 3y^2$, $z''_{xx} = 6x$, $z''_{xy} = 0$, $z''_{yy} = 6y$ xusuiy hosilalri $O(0,0)$ koordinatalar boshida nolga aylanadi, shuning uchun bu nuqtada $\Delta = 0$ bo‘ladi. Hamda $O(0,0)$ nuqtaning ixtiyoriy kichik atrofida bu funksiya har xil ishoralarni qabul qiladi, ya’ni bu nuqtada funksiya ekstremumga erishmaydi.

$z = x^4 + y^4$ funksiya va uning $z'_x = 4x^3$, $z'_y = 4y^3$, $z''_{xx} = 12x^2$, $z''_{xy} = 0$, $z''_{yy} = 12y^2$ xusuiy hosilalri $O(0,0)$ koordinatalar boshida nolga aylanadi, shuning uchun bu nuqtada $\Delta = 0$ bo‘ladi. Hamda $O(0,0)$ nuqtaning o‘zida nol, ixtiyoriy kichik atrofida esa bu funksiya faqar musbat ishorali qiymatlarni qabul qiladi, ya’ni bu nuqtada funksiya minimumga erishadi.

Shunday qilib, $\Delta = 0$ bo‘lganda funksiya ekstremumga erishishi ham, erishmasligi ham mumkin ekan.◄

7.20-Misol. $z = x^2 + y^2 - xy + x + y + 2$ funksiyaning ekstremum qiymatini toping.

►Bu yerda $z'_x = 2x - y + 1$ va $z'_y = 2y - x + 1$. Xususiy hosilalar mavjud bo‘lmagan nuqtalar yo‘q.

$$\begin{cases} 2x - y + 1 = 0 \\ -x + 2y + 1 = 0 \end{cases}$$

sistemani yechib funksiyaning kritik nuqtalarini topamiz. Bu nuqta $P(-1,-1)$ bo‘ladi.

Berilgan funksiyaning ikkinchi taribli xususiy hosilalrini topamiz:

$$z''_{xx} = 2, z''_{xy} = -1, z''_{yy} = 2.$$

Demak, $\Delta = 2 \cdot 2 - (-1)^2 = 3 > 0$ va $z''_{xx}(-1,-1) = 2$, ya’ni $P(-1,-1)$ nuqtada funksiya minimumga erishadi va $z_{min} = 1$.◄

7.11. Shartli ekstremum

Ba’zan ko‘p o‘zgaruvchili funksiya o‘zgaruvchilarining mumkin bo‘lgan o‘zgarishiga qo‘shimcha chegaralanishlar qo‘yilgan holda *funksiyaning ekstremumlarini topish* masalalari ham uchrab turadi. Bunday chegaralanishlar turli ko‘rinishlarda bo‘lishi mumkin. Masalan, o‘zgaruvchilarning qiymatlari bitta yoki bir nechta tenglama yoki tengsizliklarni qanoatlantirishi lozim bo‘ladi. Ba’zan esa o‘zgaruvchilar tekislikning ma’lum bir sohasidagina o‘zgarishi mumkin bo‘ladi. Biz quyida funksiyaning argumentlari *bog‘lanish tenglamasi* deb ataluvchi tenglamani qanoatlantirishi talab qilinadigan chegaralanishlarda to‘xtalib o‘tamiz.

7.21-Misol. Perimetri $2p$ bo‘lgan to‘g‘ri to‘rtburchaklar orasida eng katta yuzali to‘g‘ri to‘rtburchakni toping.

►x va y orqali to‘g‘ri to‘rtburchak tomonlari uzunliklarini belgilaymiz. Natijada $2(x+y)=2p$ qo‘shimcha shartda $S(x,y)=xy$ to‘g‘ri to‘rtburchak yuzining maksimumini topish masalasiga kelamiz. Uni qisqa qilib

$$S(x,y)=xy\to max,\ \ 2(x+y)=2p \qquad (7.31)$$

ko‘rinishda yozish mumkin. Albatta bu yerda yechim $x>0, y>0$ bo‘lishi kerak.

Bu holda $2(x+y)=2p$ *bog‘lanish tenglamasida* o‘zgaruvchilardan birini ikkinchisi orqali ifofalab va natijani $S(x,y)$ funksiyaning ifodasiga qo‘yib, masalaning yechimini osongina topish mumkin. Natijada bir o‘zgaruvchili funksiyaning maksimumini topish masalasiga kelamiz. Masalan, bog‘lanish tenglamasidan $y=p-x$ ifodani topamiz. U holda bu chegaralanishda to‘g‘ri to‘rtburchakning yuzi faqat x o‘zgaruvchining funksiyasiga aylanib qoladi: $S(x)=x(p-x)$. Yuqorida aytib o‘tilgan $x>0$, $y>0$ tabiiy chegaralanishdan x o‘zgaruvchining $0<x<p$ o‘zgarish sohasini topamiz. $S(x)$ funksiya $(0,p)$ oraliqda $x=p/2$ bo‘lganda maksimumga erishadi. Bu esa berilgan masalaning $x=y=p/2$ yechimini beradi. ◄

$S(x,y)=xy$ funksiyaning ekstremumi yo‘qligini ta’kidlab o‘tish kerak. Qaralgan masalada esa yechim mavjud. Buning sababi (7.31) masalada chegaralanishni qanoatlantirmaydigan nuqtalarda $S(x,y)$ funksiyaning qiymatlari bizni umuman qiziqtirmaydi, to‘g‘rirog‘i ularni biz qaramaymiz.

$z=f(x,y)$ funksiya biror D sohada aniqlangan va L bu sohada yotuvchi egri chiziq bo‘lsin. $f(x,y)$ funksiyaning L egri chiziq nuqtalariga mos keluvchi qiymatlari orasidan ekstremumlarni topish talab qilinadi. Bunday ekstremumlar $z = f(x,y)$ funksiyaning L egri chiziqdagi *shartli ekstremumi* deb ataladi.

7.22-Ta'rif. Agar L egri chiziqda yotuvchi $M_0(x_0, y_0)$ nuqtaning biror atrofidagi bu egri chiziqning ixtiyoriy $M(x,y)$ nuqtalari uchun

$$f(x,y) < f(x_0, y_0)$$

tengsizlik o'rinli bo'lsa, M_0 nuqta $f(x,y)$ funksiyaning shartli maksimum nuqtasi deyiladi.

7.23-Ta'rif. Agar L egri chiziqda yotuvchi $M_0(x_0, y_0)$ nuqtaning biror atrofidagi bu egri chiziqning ixtiyoriy $M(x,y)$ nuqtalari uchun

$$f(x,y) > f(x_0, y_0)$$

tengsizlik o'rinli bo'lsa, M_0 nuqta $f(x,y)$ funksiyaning shartli minimum nuqtasi deyiladi.

Agar L egri chiziq $\varphi(x,y) = 0$ tenglama bilan berilgan bo'lsa, $z = f(x,y)$ funksiyaning L egri chiziqdagi shartli ekstremumini topish masalasini quyidagicha ifodalash mumkin: $\varphi(x,y) = 0$ shartda $z = f(x,y)$ funksiyaning D sohadagi ekstremumini toping.

Shunday qilib, $z = f(x,y)$ funksiyaning shartli ekstremumini topishda x va y o'zgaruvchilarni erkin o'zgaruvchilar deb qarash kerak emas: ular o'zaro $\varphi(x,y) = 0$ *bog'lanish tenglamasi* bilan bog'langan.

$$z = f(x,y) \tag{7.32}$$

funksiyaning

$$\varphi(x,y) = 0 \tag{7.33}$$

shartni qanoatlantiruvchi shartli ekstremumni topish uchun (7.33) tenglamada o'zgaruvchilardan birini, masalan y o'zgaruvchini x orqali ifodalab olamiz:

$$y = \psi(x).$$

So'ngra $z = f(x,y)$ funksiyada y o'rniga $\psi(x)$ funksiyani qo'yib, x o'zgaruvchining

$$z = f\big(x, \psi(x)\big) = F(x) \tag{7.34}$$

funksiyasini hosil qilamiz. Bu funksiyada (7.33) shart inobatga olingan. Hosil qilingan $F(x)$ funksiyaning ekstremumi (shartsiz ekstremumi) qidirilayotgan shartli ekstremum bo'ladi.

7.22-Misol. $z = x^2 + y^2$ funksiyaning $x + y - 1 = 0$ to'g'ri chiziqdagi shartli ekstremumini toping.

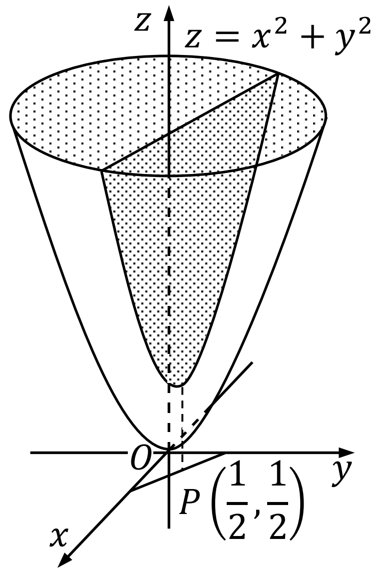
► To‘g‘ri chiziq tenglamasidan $y = 1 - x$ ifodani topamiz va uni z funksiyaga qo‘yamiz va bitta x o‘zgaruvchining

$$z = x^2 + (1 - x)^2 = 2x^2 - 2x + 1$$

funksiyasini hosil qilamiz. Uni ekstremumga tekshiramiz:

$$z' = 4x - 2,$$

uni nolga tenglashtirib $x = 1/2$ kritik nuqtani topamiz. $z'' = 4 > 0$ bo‘lganligi uchun $P(1/2\ ,1/2)$ nuqta berilgan z funksiyaning shartli minimumi bo‘ladi (7.10-rasm).◄



7.10-rasm

Biroq qo‘yilgan masalani (7.33) tenglamada o‘zgaruvchilardan birini ikkinchisi orqali ifodalamasdan ham yechish mumkin. Biz quyida *Lagranj ko‘paytuvchilari usuli* deb nomlangan usulni keltiramiz.

x o‘zgaruvching z funksiya maksimumga yoki minimumga erishadigan qiymatlarida z dan x bo‘yicha hosila nolga teng bo‘ladi. (7.32) tenglikdan murakkab funksiyaning hosilasi formulasiga ko‘ra

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

bo‘ladi. Demak ekstremum nuqtalarida

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad (7.35)$$

tenglik o‘rinli bo‘lishi kerak. (7.33) tenglamadan esa

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad (7.36)$$

tenglikni hosil qilamiz.

(7.36) tenglikning hadlarini hozircha noma’lum λ koeffisiyentga ko‘paytirib, (7.35) tenglikning mos hadlariga qo‘shsak

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} + \lambda\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx}\right) = 0$$

yoki

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y}\right) \cdot \frac{dy}{dx} = 0 \qquad (7.37)$$

tenglikka ega bo‘lamiz. $\varphi'_y \neq 0$ deb faraz qilib, λ ko‘paytuvchini

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

bo‘ladigan qilib tanlaymiz. U holda (7.37) tenglikdan xuddi shu x va y larda

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

bo‘lishi kelib chiqadi. Shunday qilib ekstremum nuqtalari uchta x, y, λ noma’lumli

$$\begin{cases} \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \\ \varphi(x, y) = 0 \end{cases} \quad (7.38)$$

uchta tenglamalar sistemasini qanoatlantirar ekan.

(7.38) sistema *Lagranj funksiyasi* deb ataluvchi

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y) \quad (7.39)$$

funksiya ekstremumining zaruriy shartlarini ifodalaydi.

Shunday qilib, $\varphi(x, y) = 0$ bo‘lganda $z = f(x, y)$ funksiyaning shartli ekstremumi

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$$

Lagranj funksiyasining kritik nuqtasi bo‘lar ekan. Bu yerdan *shartli ekstremumni topish qoidasini* hosil qilamiz:

$z = f(x, y)$ funksiyaning $\varphi(x, y) = 0$ shartni qanoatlantiruvchi shatli ekstremum nuqtasi bo‘lishi mumkin bo‘lgan nuqtalarni topish uchun:

1) $L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$ Lagranj funksiyasini tuzamiz;
2) bu funksiyaning x , y *va* λ bo‘yicha xususiy hosilalrini nolga tenglashtirib

$$\begin{cases} \frac{\partial L}{\partial x} \equiv \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \\ \frac{\partial L}{\partial y} \equiv \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} \equiv \varphi(x, y) = 0 \end{cases}$$

sistemani hosil qilamiz va bu sistemani yechib λ parametrning va ekstremum bo‘lishi mumkin bo‘lgan nuqtaning x va y koordinatalarini topamiz.

Shartli eksremumning mavjudligi va uning turini aniqlash Lagranj funksiyasi ikkinchi tartibli hosilasining ishoralarini o‘rganish orqali aniqlanadi. Lagranj funksiyasining $M_0(x_0, y_0)$ kritik nuqtasida

$$\Delta = \begin{vmatrix} 0 & \varphi'_x(M_0) & \varphi'_y(M_0) \\ \varphi'_x(M_0) & L''_{xx}(M_0) & L''_{xy}(M_0) \\ \varphi'_y(M_0) & L''_{xy}(M_0) & L''_{yy}(M_0) \end{vmatrix} \tag{7.40}$$

determinantning qiymati $\Delta > 0$ bo‘lsa, $f(x, y)$ funksiya bu nuqtada shartli maksimumga va $\Delta < 0$ bo‘lsa shartli minimumga ega bo‘ladi.

7.23-Misol. $z = x^2 - y^2$ funksiyaning $x - 2y + 3 = 0$ shartni qanoatlantiruvchi shartli ekstremumini toping.

►Biz yuqorida berilgan funksiyaning shartsiz ekstremumga ega emasligini ko‘rgan edik (7.9-rasm). Shunga qaramasdan bu funksiya $x - 2y + 3 = 0$ bog‘lanish tenglamali shartli ekstremumga ega.

Lagranj funksiyasini tuzamiz:

$$L(x, y, \lambda) = x^2 - y^2 + \lambda(x - 2y + 3).$$

Bu funksiya xususiy hosilalari nolga aylanadigan nuqtalarni topamiz:

$$\begin{cases} \dfrac{\partial L}{\partial x} \equiv 2x + \lambda = 0 \\ \dfrac{\partial L}{\partial y} \equiv -2y - 2\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} \equiv x - 2y + 3 = 0 \end{cases}$$

Bu sistemani yechib, $x = 1, y = 2$ va $\lambda = -2$ qiymatlarni hosil qilamiz. Shuning uchun $M_0(1,2)$ nuqta $L(x, y, -2)$ Lagranj funksiyasining kritik nuqtasi bo‘ladi. Lagranj funksiyasining ikkinchi va bog‘lanish funksiyasining birinchi tartibli $L''_{xx} = 2$, $L''_{yy} = -2$, $L''_{xy} = 0$, $\varphi'_x = 1, \varphi'_y = -2$ hosilalarini (7.40) determinantga qo‘yib uning ishorasini aniqlaymiz:

$$\Delta = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 2 & 0 \\ -2 & 0 & -2 \end{vmatrix} = -6 < 0.$$

Demak, $z = x^2 - y^2$ funksiya x va y o‘zgaruvchilar $x - 2y + 3 = 0$ tenglama bilan bog‘langan bo‘lsa, $M_0(1,2)$ nuqtada shartli minimumga erishar ekan va uning bu nuqtadagi qiymati $z_{min}(1,2) = 1^2 - 2^2 = -3$ bo‘ladi. ◄

7.12. Uzluksiz funksiyaning eng katta va eng kichik qiymatlari

Biror $\overline{D}$ yopiq sohada uzluksiz bo‘lgan $z = f(x, y)$ funksiyaning eng katta (kichik) qiymatini topish talab qilingan bo‘lsin. Uzluksiz ikki o‘zgaruvchili funksiyalarning 4-xossasiga ko‘ra bu sohada funksiya eng katta (kichik) qiymatni qabul qiladigan $M_0(x_0, y_0)$ nuqta mavjud. Agar M_0 nuqta D sohaning ichki qismida yotsa, f funksiya bu nuqtada maksimumga (minimumga) erishadi. Demak bu holda bizni qiziqtiradigan nuqta $f(x, y)$ funksiyaning kritik nuqtalari orasida bo‘lar ekan. Biroq $f(x, y)$ funksiya o‘zining eng katta (kichik) qiymatiga sohaning chegarasida ham erishishi mumkin. Shuning uchun $z = f(x, y)$ funksiyaning chegaralangan yopiq $\overline{D}$ sohadagi eng katta (kichik) qiymatini topish uchun sohaning ichki qismida erishiladigan barcha maksimum qiymatlar, hamda sohaning chegarasida erishiladigan eng katta (kichik) qiymatlar topiladi. Ana shu topilgan qiymatlarning eng kattasi (kichigi) $z = f(x, y)$ funksiyaning $\overline{D}$ sohadagi eng katta (kichik) qiymati bo‘ladi. Differensiallanuvchi funksiya uchun buni ko‘rsatamiz.

7.24-Misol. $z = x^2y(2 - x - y)$ funksiyaning $S: x = 0, y = 0, y + x = 6$ uchburchakdagi eng katta va eng kichik qiymatini toping.

►Eng katta va eng kichik qiymatni qabul qiladigan nuqtalar uchburchakning ichki qismida ham, chegarada ham bo‘lishi mumkin. Shohaning ichki qismidagi bunday nuqtalarni topish uchun funksiyaning xususiy hosilalarini topamiz:

$$z'_x = xy(4 - 3x - 2y),\ z'_y = x^2(2 - x - 2y)$$

va

$x = 0,\ y = 0$ hamda $\begin{cases} 3x + 2y = 4 \\ x + 2y = 2 \end{cases}$ sistemani hosil qilamiz. Sistemani yechib $P_1(1{,}1/2)$ kritik nuqtani topamiz.

$x = 0$ va $y = 0$ chegaralarda funksiya $z = 0$ qiymatni qabul qiladi.

Endi $x + y = 6$ to‘g‘ri chiziq kesmasidagi shartli kritik nuqtalarni topamiz. To‘g‘ri chiziq tenglamasidan $y = 6 - x$ ifodaga ega bo‘lamiz va uni funksiyda y o‘zgaruvchi o‘rniga qo‘yamiz:

$$z = x^2(6 - x)\big(2 - x - (6 - x)\big) = -4x^2(6 - x).$$

Uning $z'(x) = 12x(x - 4)$ hosilasini nolga tenglashtirib $P_2(0{,}6)$, $P_3(4{,}2)$ kritik nuqtalarga, hamda kesma chegarasidagi $P_4(6{,}0)$ nuqtaga ega bo‘lamiz. z funksiyaning bu to‘rtta nuqtadagi qiymatlarini hisoblaymiz:

$$z_1 = z\left(1, \frac{1}{2}\right) = \frac{1}{4}, z_2 = z(0{,}6) = 0,$$
$$z_3 = z(4{,}2) = -128, z_4 = z(6.0) = 0.$$

Ular orasida eng kattasi $1/4$ va eng kichigi -128. Shuning uchun berilgan funksiyaning S uchburchakdagi eng katta va eng kichik qiymatlari

$$z_{ekat} = \frac{1}{4}, \qquad z_{ekich} = -128$$

bo‘ladi.◄

Nazorat savollari

1. Qanday nuqtalarga to‘plamning ichki nuqtasi deb ataymiz?
2. Qanday to‘plamlar ochiq, qanday to‘plamlar yopiq to‘plam deb ataladi?
3. Qanday to‘plamlar bog‘lamli to‘plam deb ataladi?
4. Ikki o‘zgaruvchili funksiyaga ta’rif bering.
5. Ikki o‘zgaruvchili funksiyaning limitiga ta’rif bering.
6. Qanday shart bajarilganda $z = f(x, y)$ funksiya D sohada chegaralangan deyiladi?
7. Yig‘indi, ayirma va ko‘paytmaning limiti nimaga teng?
8.Qachon bo‘linmaning limiti mavjud bo‘ladi?
9. Ikki o‘zgaruvchili funksiyaning nuqtada uzluksizligiga ta’rif bering.
10. Uzluksiz funksiyalarning asosiy xossalarini sanab o‘ting.
11. Ikki o‘zgaruvchili funksiyaning xususiy va to‘la orttirmalrining farqi nimada?
12. Ikki o‘zgaruvchili funksiyaning xususiy hosilasiga ta’rif bering.
13. Ikki o‘zgaruvchili funksiya xususiy hosilasining geometrik ma’nosi nimadan iborat?
14. Ikki o‘zgaruvchili funksiyani qachon differensiallanuvchi deb ataymiz?
15. Ikki o‘zgaruvchili funksiyaning to‘la orttirmasi bilan to‘la differensiali nima bilan farq qiladi?
16. Murakkab funksiyaning xususiy hosilalri qanday topiladi?
17. Sirtga o‘tkazilgan urinma tekislik va normal tenglamasini yozing.
18. Ikkinchi tartibli xususiy hosilalar qanday topiladi?
19. Ikki o‘zgaruvchili funksiya uchun Teylor formulasini yozing.
20. Ikki o‘zgaruvchili funksiya uchun Teylor formulasi bilan Makloren formulasining farqi nimada?
21. Ikki o‘zgaruvchili funksiyaning lokal maksimum va lokal minimumiga ta’rif bering.

22. Ikki o‘zgaruvchili funksiya ekstremumining zaruriy sharti nimadan iborat?
23. Ikki o‘zgaruvchili funksiya ekstremumining yetarli sharti nimadan iborat?
24. Ikki o‘zgaruvchili funksiyaning shartli ekstremumiga ta’rif bering.
25. Lagranj funksiyasi nima?
26. Uzluksiz funksiyaning yopiq sohadagi eng katta va eng kichik qiymatlari qanday topiladi?

Mashqlar.

Quyidagi funksiyalarning aniqlanish va qiymatlar sohasini toping.

1. $z=\sqrt{1+x^2+y^2}$. **2.** $z=e^{\sqrt{x}+\sqrt{y}}$. **3.** $z=\sqrt[3]{1-x^2-y^2}$.
4. $z=\ln(x-y)$.

Limitlarni hisoblang.

5. $\lim\limits_{\substack{x\to 2\\ y\to 0}}\frac{\operatorname{tg}xy}{y}$. **6.** $\lim\limits_{\substack{x\to 3\\ y\to 7}}\lg(x+y)$. **7.** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sin xy}{x}$. **8.** $\lim\limits_{\substack{x\to \infty\\ y\to \infty}}\frac{x+y}{x^2-xy+y^2}$.

Quyidagi funksiyalarning xususiy hosilalarini va to‘la differensialini toping.

9. $z=\ln(x^2+y^2)-xy$. **10.** $z=e^{xy}+\operatorname{tg}x^2y$.
11. $z=x\sqrt{y}-2\sqrt{x^2+y^2}$.

Funksiyaning berilgan nuqtadagi xususiy hosilalarini hisoblang.

12. $z=\sqrt{x^2+y^2}$, $M(3;4)$. **13.** $z=\ln xy\,, M(1;2)$.
14. $z=e^{x-y}, M(3;1)$.
15. $z=x^3+y^2$ sirtga $P(1;2;5)$ nuqtada o‘tkazilgan urinma va normal tenglamasini tuzing.
16. $1{,}08^{3{,}96}$ ifodaning qiymatini taqribiy hisoblang.

Funksiyaning ikkinchi tartibli xususiy hosilalarini toping**.**

17. $z=\sin(x-y^2)$. **18.** $z=x^2y-5x+4y-1$. **19.** $z=e^{x+y}$.
20. $z=e^{xy}$ funksiyaning z'''_{y^2x} xususiy hosilasini toping.
21. $z=x/y$ funksiyaning uchinchi tartibli d^3z differensialini toping.
22. $z=x^y$ funksiyani $M_0(1;1)$ nuqtaning atrofida $n=2$ hol uchun Teylor formulasi bo‘yicha yoying va $1{,}1^{1{,}02}$ ifodaning qiymatini taqribiy hisoblang.

Funksiyani ekstremumga tekshiring.

23. $z=x^3+y^3-3xy$. **24.** $z=x^2-xy+y^2+9x-6y+1$.
25. $z=2xy-3x^2-2y^2+10$. **26.** $z=4(x-y)-x^2-y^2$.
27. $z=e^{x^2-y^2}(2x^2+y^2)$.

Funksiyani berilgan shart boyicha shartli ekstremumga tekshiring.

28. $z = x^2 + (y-2)^2$, $x^2 - y^2 - 4 = 0$. **29.** $z = \frac{1}{x} + \frac{1}{y}$, $x + y - 2 = 0$.
30. $z = 2x^3 - 6xy + 3y^2$ funksiyaning Oy o‘q, $y = 2$ to‘g‘ri chuiziq va $y = \frac{x^2}{2}$ parabolaning $x \geq 0$ bo‘lgandagi qismi bilan chegaralangan sohadagi eng katta va eng kichik qymatlarini toping.
31. $z = x^2y(2 - x - y)$ funksiyaning $x = 0$, $y = 0$, $y + x = 6$ to‘g‘ri chiziqlar bilan chegalangan uchburchakdagi eng katta va eng kichik qymatlarini toping.

Javoblar

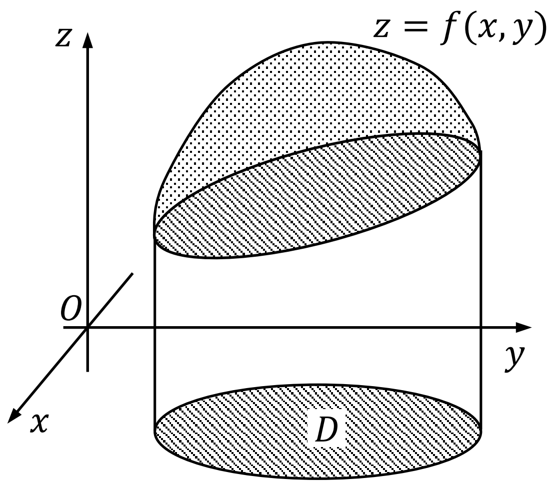
1. Oxy tekislikning barcha nuqtalari. **2.** I chorak. **3.** Oxy tekislikning barcha nuqtalari. **4.** $x > y$ yarim tekislik. **5.** 2. **6.** 1. **7.** 0. **8.** 0.
9. $z'_x = 2x/(x^2+y^2) - y$, $z'_y = 2y/(x^2+y^2) - x$,
$$dz = \left(\frac{2x}{x^2+y^2} - y\right)dx + \left(\frac{2y}{x^2+y^2} - x\right)dy.$$
10. $z'_x = ye^{xy} + 2xy/\cos^2 x^2y$, $z'_y = xe^{xy} + x^2/\cos^2 x^2y$,
$dz = (ye^{xy} + 2xy/\cos^2 x^2y)dx + (e^{xy} + x^2/\cos^2 x^2y)dy$.
11. $z'_x = \sqrt{y} - 2x/\sqrt{x^2+y^2}$, $z'_y = x/2\sqrt{y} - 2y/\sqrt{x^2+y^2}$,
$$dz = \left(\sqrt{y} - \frac{2x}{\sqrt{x^2+y^2}}\right)dx + \left(\frac{x}{2\sqrt{y}} - \frac{2y}{\sqrt{x^2+y^2}}\right)dy.$$
12. $z'_x(M) = 3/5, z'_y(M) = 4/5$. **13.** $z'_x(M) = 1, z'_y(M) = 1/2$.
14. $z'_x(M) = e^2, z'_y(M) = -e^2$. **15.** $z - 5 = 3(x-1) + 4(y-4)$. **16.** 1,32. **17.** $z''_{xx} = -\sin(x - y^2), z''_{xy} = 2y\sin(x - y^2), z''_{yy} = -2\cos(x - y^2) - 4y^2\sin(x - y^2)$. **18.** $z''_{xx} = 2y, z''_{xy} = 2x, z''_{yy} = 0$. **19.** $z''_{xx} = e^{x+y}, z''_{xy} = e^{x+y}, z''_{yy} = e^{x+y}$. **20.** $z'''_{y^2x} = (2 + xy)xe^{xy}$.
21. $d^3z = \frac{6}{y^3}dxdy^2 - \frac{6x}{y^4}dy^3$. **22.** $x^y \approx 1 + (x-1) + (x-1)(y-1)$, $1{,}1^{1{,}02} \approx 1{,}102$. **23.** $z_{min} = z(1;1) = -1$. **24.** $z_{min} = z(-4;1) = -20$. **25.** $z_{max} = z(0;0) = 10$. **26.** $z_{max} = z(2;-2) = 8$. **27.** $z_{min} =$
$= z(0;0) = 0, z_{max} = z(\pm 1; 0) = 2e$. **28.** $z_{min} = z(\pm\sqrt{5}; 1) = \sqrt{6}$. **29.** $z_{min} = z(1;1) = 2$. **30.** $z_{ekat} = z(0;2) = 12$, $z_{ekich} = z(1;1) = -1$.
31. $z_{ekat} = z(0;1/2) = 1/4, z_{ekich} = z(4;2) - 128$.

VIII BOB. IKKI VA UCH O‘LCHOVLI INTEGRALLAR

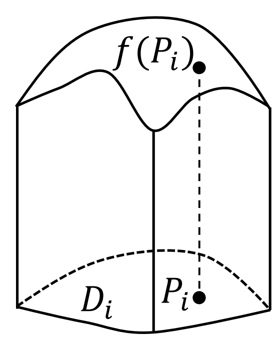
8.1. Ikki o‘lchovli integral tushunchasiga keltiriladigan masala

Egri chiziqli trapetsiyaning yuzini hisoblash masalasi aniq integral tushunchasiga olib kelgan edi. Ikki o‘lchovli integral tushunchasiga silindrik jismning hajmini hisoblash masalasini yechish orqali kelamiz.

$Oxyz$ to‘g‘ri burchakli koordinatalar sistemasida quyidan Oxy tekislikning D yopiq sohasi bilan, yon tomonlardan Ozo‘qiga parallel silindrik sirt bilan va yuqoridan $z = f(x, y)$ tenglama bilan aniqlanuvchi sirt bilan chegaralangan Q jismni qaraymiz (8.1-rasm). Bu yerda $(x; y) \in D$ nuqtalarda $f(x, y)$ funksiya nomanfiy qiymatlarni qabul qiladi deb faraz qilamiz. Bunday tasvirlangan V jismni *silindrik jism*, x va y o‘zgaruvchilarning D o‘zgarish sohasi esa silindrik jismning *asosi* deb ataladi.



8.1-rasm



8.2-rasm

Qaralayotgan Q jismning V hajmini topish uchun uning D asosini ixtiyoriy chiziqlar bilan o‘zaro kesishmaydigan n ta D_i qismiy sohalarga ajratamiz. U holda Q jismni asoslari D_i sohalardan iborat bo‘lgan n ta $Q_i, i = \overline{1, n}$ silindrik ustunning yig‘indisi deb qarash mumkin. Har bir qismiy sohada birorta $P_i(\xi_i, \eta_i) \in D_i$ nuqtani tanlaymiz (8.2-rasm). Agar har bir silindrik ustunni balandligi $f(\xi_i, \eta_i)$ bo‘lgan to‘g‘ri silindr bilan almashtirsak, Q_i ustunning ΔV_i hajmi taqriban $f(\xi_i, \eta_i)\Delta S_i$ ko‘paytmaga teng bo‘ladi, bu yerda ΔS_i qismiy D_i sohaning yuzi. U holda Q jismning yuzi taqriban

$$V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta S_i \qquad (8.1)$$

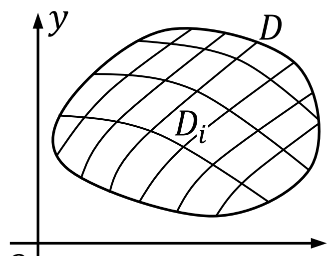
yig‘indiga teng bo‘ladi.

Bu munosabatning aniqligini oshirish uchun D_i qismiy sohalarning sonini oshirish hisobiga ularning o‘lchamlarini kamaytirish kerakligi ravshan. Hajmning V aniq qiymati sifatida (8.1) yig‘indining $D_i, i = \overline{1,n}$ to‘plamlar *eng katta d diametrining* nolga intilgandagi limitini olish kerak:

$$V = \lim_{d\to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{d\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta S_i \qquad (8.2)$$

8.2. Ikki o‘lchovli integral

Asosiy tushuncha va ta’riflar. Agar chiziqning boshlang‘ich va oxirgi nuqtasi ustma-ust tushsa, biz uni yopiq chiziq deb ataymiz. Yopiq chiziq bilan chegaralanagan to‘plamga chegaralovchi chiziq ham tegishli bo‘lsa, bunday to‘plamlar yopiq to‘plam deb ataladi.



8.3-rasm

Oxy tekislikning D sohasida uzluksiz $z = f(x, y)$ funksiya aniqlangan bo‘lsin. D sohani biror usul bilan n ta $D_i, i = \overline{1,n}$ qismiy sohalarga ajratamiz va ularning yuzlarini mos ravishda ΔS_i, diametrlarini esa d_i va bu diametrlarning eng kattasini d orqali belgilaymiz (8.3-rasm).

Har bir D_i sohada yxtiyoriy ravishda $M_i(\xi_i; \eta_i)$ nuqtani tanlaymiz va

$$f(\xi_1, \eta_1)\Delta S_1 + f(\xi_1, \eta_1)\Delta S_1 + \cdots + f(\xi_n, \eta_n)\Delta S_n = \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta S_i \qquad (8.3)$$

yig‘indini tuzamiz. Bu yig‘indi $f(x, y)$ funksiyaning D soha bo‘yicha *integral yig‘indisi* deb ataladi.

8.1-Ta’rif. Agar (8.3) integral yig‘indi $d = 0$ nuqtada chekli I limitga ega bo‘lib, u D sohani bo‘lish usuliga va undagi nuqtalarning tanlanishiga bog‘liq bo‘lmasa, bu limitni biz $f(x, y)$ funksiyaning D soha bo‘yicha olingan ikki o‘lchovli integrali deb ataymiz va uni

$$I = \iint_D f(x, y)dS = \iint_D f(x, y)dxdy$$

orqali belgilaymiz.

Shunday qilib, ikki o‘lchovli integral

$$\iint_D f(x, y)dxdy = \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta S_i \qquad (8.4)$$

tenglik bilan aniqlanar ekan. Bunday holda $f(x, y)$ funksiya D sohada *integrallanuvchi* va D *−integrallash sohasi* deb ataladi.

(8.2) va (8.4) tengliklarni taqqoslab, 8.1. da qaralgan hajm

$$V = \iint\limits_D f(x,y)dxdy \qquad (8.5)$$

formula bilan hisoblanadi deb xulosa chiqarish mumkin.

$f(x,y)$ funksiya D sohada chegaralangan bo‘lsin. U holda $f(x,y)$ har bir D_i $(i = 1,2,\dots,n)$ sohada ham chegaralangan bo‘ladi. $m_i = \inf\limits_{(x,y)\in\Delta D_i} f(x,y)$, $M_i = \sup\limits_{(x,y)\in\Delta D_i} f(x,y)$ deb belgilash kiritamiz va $f(x,y)$ funksiyaning D sohadagi yuqori va quyi Darbu yig‘indilari deb ataluvchi

$$\overline{S} = \sum_{i=1}^{n} M_i \Delta S_i,\ \underline{S} = \sum_{i=1}^{n} m_i \Delta S_i$$

yig‘indilarni tuzamiz. Bu yig‘indilar ham bir o‘zgaruvchili funksiyaning Darbu yig‘indilari ega bo‘lgan barcha xossalarga ega.

Har qanday $f(x;y)$ funksiya uchun ham ikki o‘lchovli integral mavjudmi degan savolning tug‘ilishi tabiiy. Ikki o‘lchovli integral mavjudligining yetarli shartini topish uchun, bir o‘zgaruvchili funksiyalardagi singari yuqorida kiritilgan Darbu yig‘indilaridan foydalanish juda qulay va ular ikki o‘lchovli integrallar uchun to‘liq o‘tadi. Quyidagi teorema ham aniq integraldagi mos teorema singari isbotlanadi.

8.1-Teorema. Yopiq chegaralanagan D sohada uzluksiz $f(x,y)$ funksiya, bu sohada integrallanuvchi hamdir.

Biz quyida integrallash sohasida faqat uzluksiz bo‘lgan funksiyalarnigina qaraymiz.

8.3. Ikki olchovli integralning xossalari

Ikki o‘lchovli integralning xossalari aniq integralning mos xossalari bilan bir xil va isbotalri ham shu tartibda takrorlanadi. Shuning uchun ayrim xossalarning isbotidagina to‘xtalamiz.

1) Agar D soha S yuzaga ega bo‘lsa, u holda

$$\iint\limits_D dxdy = S.$$

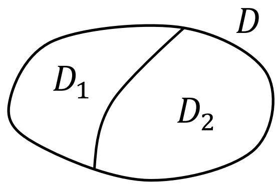
►Bu holda integral osti funksiya D sohada o‘zgarmas: $f(x,y) \equiv 1$. Shuning uchun D sohani yuzalari ΔS_i bo‘lgan$D_i, i = \overline{1,n}$ sohalarga ixtiyoriy bo‘lishlarda va $(\xi_i, \eta_i) \in D_i$ nuqtalarni ixtiyoriy tanlashda ikki o‘lchovli in tegralning ta’rifiga ko‘ra

$$\iint\limits_D dxdy = \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{i=1}^{n} 1\cdot\Delta S_i = S.◄$$

2) Agar $f(x,y)$ va $g(x,y)$ funksiyalar D sohada integrallanuvchi bo‘lsa, ularning $\alpha f(x,y)+\beta g(x,y)$ kombinatsiyasi ham D sohada integrallanuvchi va

$$\iint_D (\alpha f(x,y)+\beta g(x,y))dxdy =$$
$$= \alpha\iint_D f(x,y)dxdy + \beta\iint_D g(x,y)dxdy, \qquad (8.6)$$

tenglik o‘rinli bo‘ladi bu yerda $\alpha,\beta \in \mathbb{R}$.



8.4-rasm

3) Agar D sohani $D_1 \cup D_2 = D$ bo‘ladigan qilib faqat bo‘linish chizig‘idagina kesishadigan ikkita D_1 va D_2 sohalarga ajratilsa (8.4-rasm),

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy \qquad (8.7)$$

tenglik o‘rinli bo‘ladi.

4) Agar $f(x,y)$ funksiya D sohada nomanfiy va integrallanuvchi bo‘lsa,

$$\iint_D f(x,y)dxdy \geq 0 \qquad (8.8)$$

bo‘ladi.

5) Agar $f(x,y)$ va $g(x,y)$ funksiyalar D sohada integrallanuvchi va barcha $(x,y) \in D$ nuqtalarda $f(x,y) \geq g(x,y)$ bo‘lsa

$$\iint_D f(x,y)dxdy \geq \iint_D g(x,y)dxdy \qquad (8.9)$$

bo‘ladi.

6) Agar $f(x,y)$ funksiya D sohada integrallanuvchi bo‘lsa, $|f(x,y)|$ funksiya ham D sohada integrallanuvchi bo‘ladi va

$$\left|\iint_D f(x,y)dxdy\right| \geq \iint_D |f(x,y)|dxdy \qquad (8.10)$$

tengsizlik o‘rinli bo‘ladi.

7) Agar yuzasi S ga teng bo‘lgan D sohada $f(x,y)$ funksiya uzluksiz va uning bu sohadagi eng katta va eng kichik qiymatlari mos ravishda M va m bo‘lsa,

$$mS \leq \iint_D f(x,y)dxdy \leq MS \qquad (8.11)$$

tengsizliklar o‘rinli bo‘ladi.

8) (O‘rta qiymat haqidagi teorema) Agar yuzasi S ga teng bo‘lgan D sohada $f(x,y)$ funksiya uzluksiz bo‘lsa, bu sohada

$$\iint_D f(x,y)dxdy = f(x_0,y_0)\cdot S$$

tenglikni qanoatlantiruvchi $P_0(x_0,y_0)$ nuqta mavjud.

$$f(x_0,y_0) = \frac{1}{S}\iint_D f(x,y)dxdy \qquad (8.12)$$

miqdor $f(x,y)$ funksiyaning D sohadagi *o‘rta qiymati* deb ataladi.

8.4. Ikki o‘lchovli integralni hisoblash

Ikki o‘lchovli integralni to‘g‘ri t o‘rtburchak soha bo‘yicha hisoblash. Dastlab D integrallash sohasi tomonlari koordinata o‘qlariga parallel bo‘lgan to‘g‘ri to‘rtburchakdan iborat sodda holni qarab chiqamiz.

8.2-Teorema. $f(x,y)$ funksiya $x=a$, $x=b$, $y=c$, $y=d$ to‘g‘ri chiziqlar bilan chegaralangan D to‘g‘ri to‘rtburchakda aniqlangan va

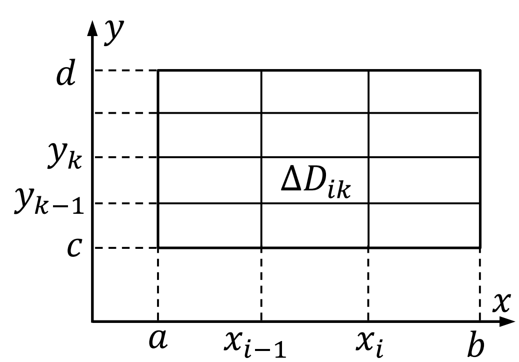
1) $I=\iint_D f(x,y)dxdy$ ikki o‘lchovli integral mavjud;

2) Ixtiyoriy fiksirlangan $x\in[a,b]$ qiymatda $\varphi(x)=\int_c^d f(x,y)dy$ aniq integral mavjud bo‘lsin. U holda

$$\int_a^b \varphi(x)dx = \int_a^b\left[\int_c^d f(x,y)dy\right]dx$$

integral ham mavjud va uning qiymati I ga teng bo‘ladi.

► $[a,b]$ kesmani x_1, x_2 ,…, x_{n-1} nuqtalar yordamida n qismga, $[c,d]$ kesmani esa y_1, y_2 ,…, y_{m-1} nuqtalar yordamida m qismga ajratamiz va $x_0=a$, $x_n=b$, $y_0=c$, $y_m=d$ deb olamiz (8.5-rasm). Bundan tashqari $\Delta x_i = x_i - x_{i-1}$, $\Delta y_k = y_k - y_{k-1}$ deb olamiz. $x=x_i$ $(i=1,2,\dots,n-1)$ va $y=y_k$ $(k=1,2,\dots,m-1)$ to‘g‘ri chiziqlarni o‘tkazib D sohani mn ta kichik to‘g‘ri to‘rtburchaklarga ajratamiz. ΔD_{ik} –ulardan biri bo‘lsin.



8.5-rasm

$$m_{ik} = \inf_{(x,y)\in\Delta D_{ik}} f(x,y),$$

$$M_{ik} = \sup_{(x,y)\in\Delta D_{ik}} f(x,y)$$

deb belgilash kiritamiz. U holda ixtiyoriy $(x,y) \in \Delta D_{ik}$ nuqtalarda $m_{ik} \leq f(x,y) \leq M_{ik}$ tengsizlik o‘rinli bo‘ladi. Xususiy holda ixtiyoriy $x = \xi_i \in [x_{k-1}, x_k]$va ixtiyoriy $y \in [y_{k-1}, y_k]$ uchun

$$m_{ik} \leq f(\xi_i, y) \leq M_{ik}$$

bo‘ladi. Bu tengsizlikni y bo‘yicha $[y_{i-1}, y_i]$ kesmada integrallab

$$m_{ik} \cdot \Delta y_k \leq \int_{y_{k-1}}^{y_k} f(\xi_i, y)dy \leq M_{ik} \cdot \Delta y_k \qquad (8.13)$$

tengsizlikka ega bo‘lamiz. Bu tengsizlikdagi integral teoremaning 2) shartiga ko‘ra mavjud. (8.13) tengsizlikni k bo‘yicha 1 dan m gacha hadma-had qo‘shib chiqsak

$$\sum_{k=1}^{m} m_{ik} \cdot \Delta y_k \leq \int_{c}^{d} f(\xi_i, y)dy \leq \sum_{k=1}^{m} M_{ik} \cdot \Delta y_k$$

tengsizlik hosil bo‘ladi va unga teorema shartidagi belgilashni qo‘llab

$$\sum_{k=1}^{m} m_{ik} \cdot \Delta y_k \leq \varphi(\xi_i) \leq \sum_{k=1}^{m} M_{ik} \cdot \Delta y_k$$

tengsizlikni yozamiz. Bu tengsizlikni Δx_i kesma uzunligiga ko‘paytiramiz va so‘ngra i bo‘yicha 1 dan n gacha hadma-had qo‘shamiz:

$$\sum_{i=1}^{n} \left(\sum_{k=1}^{m} m_{ik} \cdot \Delta y_k \right) \cdot \Delta x_i \leq \sum_{i=1}^{n} \varphi(\xi_i) \cdot \Delta x_i \leq \sum_{i=1}^{n} \left(\sum_{k=1}^{m} M_{ik} \cdot \Delta y_k \right) \cdot \Delta x_i.$$

$\Delta x_i \cdot \Delta y_k$ ko‘paytma D_{ik} sohaning yuziga teng bo‘lganligi uchun, bu tengsizlikning chetki hadlari $f(x,y)$ funksiyaning D sohadagi quyi va yuqori Darbu yig‘indisini beradi, ya’ni bu tengsizlikni

$$\underline{S} \leq \sum_{i=1}^{n} \varphi(\xi_i) \cdot \Delta x_i \leq \overline{S}$$

ko‘rinishda yozish mumkin.

Δx_i va Δy_k bir vaqtda nolga intilsin. U holda teoremaning 1) shartidan $\underline{S} \to I$ va $\overline{S} \to I$ ekanligi va demak $\sum_{i=1}^{n} \varphi(\xi_i) \cdot \Delta x_i \to I$ ekanligi kelib chiqadi. Biroq so‘ngi yig‘indining limiti $\int_a^b \varphi(x)dx$ integralni aniqlaydi. Shunday qilib $\int_a^b \varphi(x)dx = I$ ekanligini, ya’ni

$$\int_a^b \left[\int_c^d f(x,y)dy\right] dx = \iint_D f(x,y)dxdy$$
tenglikni isbotladik.

Chap tomondagi integralda dastlab y bo‘yicha integrallanadi, so‘ngra esa x bo‘yicha integrallanadi. Shu sababli ular *takroriy integral* deb ataladi. Bu tenglikni

$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

ko‘rinishda yozsak, ikki o‘lchovli integralni hisoblash takroriy integralni hisoblashga keltirilganligini ko‘ramiz. So‘ngi munosabat ko‘pincha

$$\iint_D f(x,y)dxdy = \int_a^b dx \int_c^d f(x,y)dy \qquad (8.14)$$

ko‘rinishda yoziladi.

Izoh. Agar 1) va 2) shartlarga qo‘shimcha ravishda quyidagi shart bajarilsa:

3) Ixtiyoriy fiksirlangan $y \in [c,d]$ qiymatda $\psi(y) = \int_c^d f(x,y)dx$ aniq integral mavjud bo‘lsin, u holda

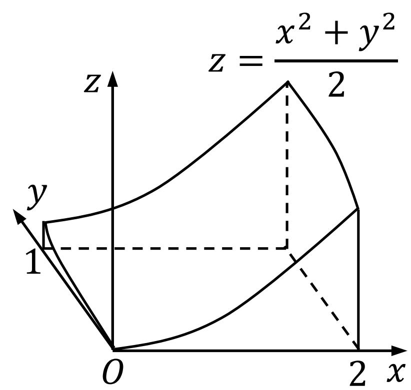
$$\int_c^d \psi(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

takroriy integral ham mavjud bo‘ladi va uning ham qiymati I ga teng bo‘ladi. Demak (8.14) formulaga teng kuchli

$$\iint_D f(x,y)dxdy = \int_c^d dy \int_a^b f(x,y)dx \qquad (8.15)$$

formula o‘rinli bo‘lar ekan.

8.1-Misol. $z = (x^2 + y^2)/2$ sirt va $x = 0$, $x = 2$, $y = 0$, $y = 1$, $z = 0$ tekisliklar bilan chegaralangan jismning hajmini toping (8.6-rasm).



8.6-rasm

►(8.5) va (8.14) formulalarga ko‘ra

$$V = \iint_D \frac{(x^2+y^2)}{2} dxdy =$$
$$= \int_0^2 dx \int_0^1 \frac{(x^2+y^2)}{2} dy =$$

$$=\frac{1}{2}\int\limits_0^2\left(x^2\cdot y+\frac{y^3}{3}\right)\Bigg|_0^1 dx=\frac{1}{2}\int\limits_0^2\left(x^2+\frac{1}{3}\right)dx=\frac{1}{2}\left(\frac{x^3}{3}+\frac{x}{3}\right)\Bigg|_0^2=\frac{1}{2}\left(\frac{8}{3}+\frac{2}{3}\right)=\frac{5}{3}.$$

Izlanayotgan jismning hajmini hosil qildik. ◄

Ikki o‘lchovli integralni ixtiyoriy soha bo‘yicha hisoblash. Endi integrallash sohasining chegarasi to‘liq yoki qisman egri chiziq bo‘lgan murakkab holni qaraymiz.

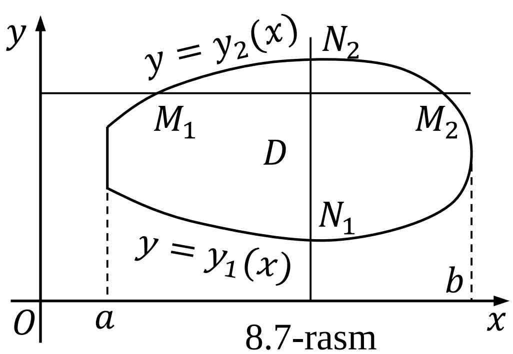
8.2-Ta’rif. D integrallash sohasini

$$D=\{(x,y)\in\mathbb{R}^2: x\in[a,b], y_1(x)\le y\le y_2(x)\} \qquad (8.16)$$

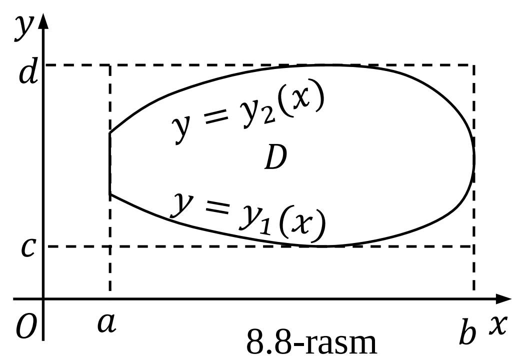
ko‘rinishda ifodalsh mumkin bo‘lsa, uni *Oy o‘q yo‘nalishi bo‘yicha to‘g‘ri soha* deb ataymiz, bu yerda $y_1(x)$ va $y_2(x)$ funksiyalar $[a,b]$ kesmada uzluksiz va $y_1(x)\le y_2(x)$ tengsizlikni qanoatlantiradi (8.7-rasm).

Bunday sohalarning ichki nuqtasi (ya’ni chegara nuqtasi bo‘lmagan) orqali Oy o‘qqa parallel qilib o‘tkazilgan har qanday to‘g‘richiziq sohaning chegarasini faqat ikki nuqtada kesib o‘tadi. 8.7-rasmda M_1 va M_2 nuqtalar. *Ox o‘q yo‘nalishi bo‘yicha to‘g‘ri soha* ham xuddi shu singari aniqlanadi.

Bir vaqtning o‘zida ham Ox va ham Oy o‘qlar yo‘nalishi bo‘yicha to‘g‘ri sohani oddiy qilib *to‘g‘ri soha* deb ataymiz. 8.7-rasmda to‘g‘ri soha tasvirlangan.



8.7-rasm



8.8-rasm

8.3-Teorema. Agar $f(x,y)$ funksiyaning (8.16) ko‘rinishdagi D soha bo‘yicha ikki o‘lchovli integrali mavjud va har bir fiksirlangan $x\in[a,b]$ qiymatda $\varphi(x)=\int\limits_{y_1(x)}^{y_2(x)} f(x,y)dy$ aniq integral mavjud bo‘lsin. U holda

$$\int\limits_a^b\varphi(x)dx=\int\limits_a^b\left[\int\limits_{y_1(x)}^{y_2(x)} f(x,y)dy\right]dx$$

integral ham mavjud va uning qiymati I ga teng bo‘ladi:

$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} f(x,y)dy. \tag{8.17}$$

►D integrallash sohasini

$$T = \{(x,y) \in \mathbb{R}^2 : x \in [a,b], y \in [c,d]\}$$

to‘g‘ri to‘rtburchak ichiga joylashtiramiz, bu yerda $[c,d]$ kesmaning c chetki nuqtasi $y_1(x)$ funksiyaning $[a,b]$ kesmadagi eng kichik qiymati, d chetki nuqta esa $y_2(x)$ funksiyaning bu kesmadagi eng katta qiymatidan iborat (8.8-rasm).

T to‘g‘ri to‘rtburchakda

$$F(x,y) = \begin{cases} f(x,y), & x \in D; \\ 0, & x \in T \backslash D \end{cases}$$

funksiyani aniqlaymiz. $f(x,y)$ funksiya D sohada integrallanuvchi bo‘lganligi sababli $F(x,y)$ funksiya ham D sohada integrallanuvchi va bundan tashqari

$$\iint\limits_D F(x,y)dxdy = \iint\limits_D f(x,y)dxdy.$$

$T \backslash D$ to‘plamda $F(x,y) = 0$ bo‘lganligi uchun $\iint\limits_{T \backslash D} F(x,y)dxdy = 0$ bo‘ladi. U holda ikki o‘lchovli integralning 3) xossasiga ko‘ra $F(x,y)$ funksiya T to‘g‘ri to‘rtburchakda integrallanuvchi va

$$\iint\limits_T F(x,y)dxdy = \iint\limits_D F(x,y)dxdy + \iint\limits_{T \backslash D} F(x,y)dxdy =$$
$$= \iint\limits_D F(x,y)dxdy = \iint\limits_D f(x,y)dxdy \tag{8.18}$$

tenglik o‘rinli bo‘ladi.

Har bir fiksirlangan $x \in [a,b]$ qiymatda $F(x,y)$ funksiyaning $[c,d]$ kesma bo‘yicha aniq integrali mavjud va

$$\int\limits_c^d F(x,y)dy = \int\limits_c^{y_1(x)} F(x,y)dy + \int\limits_{y_1(x)}^{y_2(x)} F(x,y)dy + \int\limits_{y_2(x)}^{d} F(x,y)dy.$$

Haqiqatdan ham, $[c, y_1(x)]$va $[y_2(x), d]$ kesmalarda $F(x,y)$ funksiya aynan nol va demak integrallanuvchi.

$$\int\limits_{y_1(x)}^{y_2(x)} F(x,y)dy = \int\limits_{y_1(x)}^{y_2(x)} f(x,y)dy$$

integral esa teorema shartiga ko‘ra mavjud. Aniq integralning 5) xossasiga (additivlik) binoan $F(x,y)$ funksiya $[c,d]$ kesmada integrallanuvchi va

$$\int_c^d F(x,y)dy = \int_{y_1(x)}^{y_2(x)} f(x,y)dy \tag{8.19}$$

degan xulosaga kelamiz. Bu xulosadan $F(x,y)$ funksiyaning T to‘g‘ri to‘rtburchakda 8.2-Teoremaning shartlarini qanoatlantirishi kelib chiqadi. Shuning uchun bu funksiyaning T to‘g‘ri to‘rtburchak bo‘yicha ikki o‘lchovli integrali mavjud va u takroriy integralga teng:

$$\iint_T F(x,y)dxdy = \int_a^b dx \int_c^d F(x,y)dy.$$

Bu yerda (8.18) va (8.19) tengliklarni qo‘llab, (8.17) tenglikni hosil qilamiz.◄

Izoh. D integrallash sohasi chap va o‘ngdan $x = x_1(y)$ va $x = x_2(y)$ chiziqlar bilan, quyi va yuqoridan esa $y = c$ va $y = d$ to‘g‘ri chiziqlar bilan chegaralangan bo‘lsin (8.9-rasm) va $f(x,y)$ funksiyaning D soha bo‘yicha ikki o‘lchovli integrali mavjud, hamda har bir fiksirlangan $y \in [c,d]$ qiymatda $\psi(y) = \int_{x_1(y)}^{x_2(y)} f(x,y)dx$ aniq integral mavjud bo‘lsin. U holda

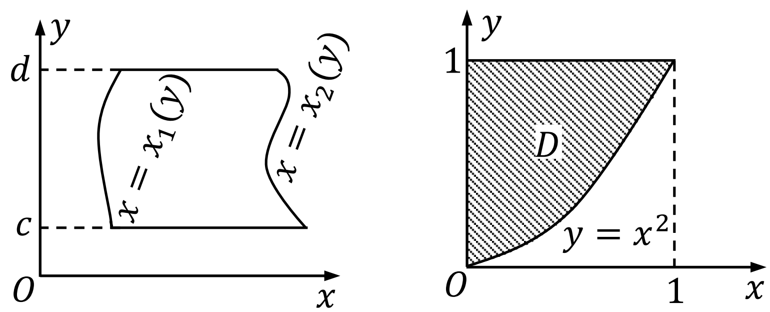
$$\int_c^d \psi(y)dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)dx\right] dy$$

integral ham mavjud va uning qiymati I ga teng bo‘ladi:

$$\iint_D f(x,y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)dx. \tag{8.20}$$

Bu tasdiq ham 8.3-Teorema singari isbotlanadi.

8.2-Misol. $f(x,y) = x^2y^3$ funksiyaning $D = \{(x,y) \in \mathbb{R}^2 : x \in [0,1], x^2 \le y \le 1\}$ soha bo‘yicha ikki olchovli integralini hisoblaymiz.



8.9-rasm 8.10-rasm

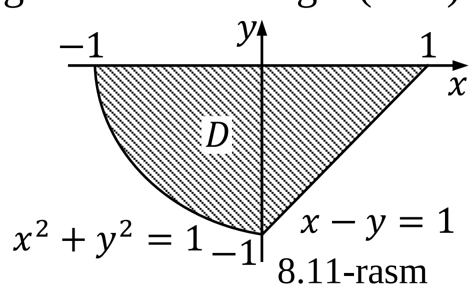
►Integrallash sohasi Oyo‘q yo‘nalishi bo‘yicha to‘g‘ri soha va $y_1(x) = x^2$ va $y_2(x) \equiv 1$ (8.10-rasm). (8.17) formulaga ko‘ra ikki o‘lchovli integralni hisoblab,

$$\iint_D x^2y^3dxdy = \int_0^1 dx\int_{x^2}^1 x^2y^3dy = \int_0^1 \frac{1}{4}x^2y^4\bigg|_{x^2}^1 dx =$$
$$= \frac{1}{4}\int_0^1 (x^2 - x^{10})dx = \frac{1}{4}\left(\frac{x^3}{3} - \frac{x^{11}}{11}\right)\bigg|_0^1 = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{11}\right) = \frac{2}{33}$$
qiymatga ega bo‘lamiz.

Mulohaza. 8.2-Misolda integrallash sohasi ikkala koordinatalar o‘qiga nisbatan ham to‘g‘ri, shuning uchun ikki o‘lchovli integralni hisoblashda (8.20) formuladan ham foydalanish mumkin. Bunda integrallash sohasini
$$D = \left\{(x,y) \in \mathbb{R}^2 \colon y \in [0,1], 0 \le y \le \sqrt{y}\right\}$$
ko‘rinishda yozib olamiz va (8.20) formuladan foydalanamiz:
$$\iint_D x^2y^3dxdy = \int_0^1 dy\int_0^{\sqrt{y}} x^2y^3dx = \int_0^1 \frac{1}{3}x^3y^3\bigg|_0^{\sqrt{y}} dy = \frac{1}{3}\int_0^1 y^{9/2}dy =$$
$$= \frac{1}{3}\cdot\frac{2}{11}y^{\frac{11}{2}}\bigg|_0^1 = \frac{2}{33}$$
va oldingi natijaga kelamiz.◄

Mulohaza. Agar ikki ikki o‘lchovli integralni hisoblash (8.17) formula bilan ham, (8.20) formula bilan ham takroriy integralga keltirilishi mumkin bo‘lsa, misolga qarab formulani tanlash kerak bo‘ladi.

8.3-Misol. $f(x,y) = 2x + 3y$ funksiyaning Ox o‘q, $x^2 + y^2 = 1$ aylananing bir qismi va $x - y = 1$ chiziqlar bilan chegaralangan D soha (8.11-rasm) bo‘icha ikki o‘lchovli integralni hisoblashga (8.17) va (8.20) formulalarning ikkalasini ham qo‘llash mumkin, chunki D soha ikkala koordinata o‘qlariga nisbatan ham to‘g‘ri. Shuning uchun (8.17) formula qo‘llanilsa, ikkita takroriy integralni hisoblashga to‘g‘ri keladi:



8.11-rasm

$$\iint_D (2x+3y)dxdy = \int_{-1}^0 dx\int_{-\sqrt{1-x^2}}^{0}(2x+3y)dy + \int_0^1 dx\int_{x-1}^{0}(2x+3y)dy.$$
Agar (8.20) formulani qo‘llasak bitta takroriy integralni hisoblash bilan chegaralanish mumkin:

$$\iint\limits_D (2x+3y)dxdy = \int\limits_{-1}^{0} dy \int\limits_{-\sqrt{1-y^2}}^{1+y} (2x+3y)dx =$$

$$= \int\limits_{-1}^{0} (x^2+3yx)\Big|_{-\sqrt{1-y^2}}^{1+y} dy =$$

$$= \int\limits_{-1}^{0} \left(1+2y+y^2+3y+3y^2-1+y^2+3y\sqrt{1-y^2}\right)dy =$$

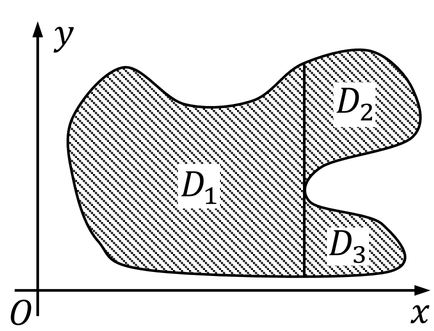
$$= \left(\frac{5}{3}y^3+\frac{5}{2}y^2-(1-y^2)^{3/2}\right)\Big|_{-1}^{0} = -1+\frac{5}{3}-\frac{5}{2} = -\frac{11}{6},$$

ya’ni (8.20) formulani qo‘llash qulay ekan.◄

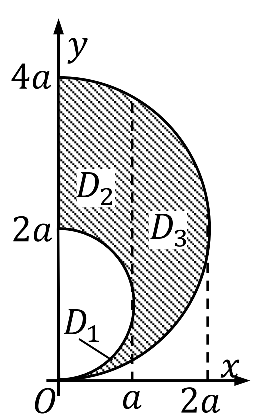
Mulohaza. Agar D integrallash sohasi Oxo‘qi yo‘nalishi boyicha ham, Oyo‘qi yo‘nalishi boyicha ham to‘g‘ri bo‘lmasa (8.8-rasm), ikki o‘lchovli integralni takroriy integralga keltirish uchun D integrallash sohasini har biri hech bo‘lmaganda bitta o‘q yo‘nalishi bo‘yicha to‘g‘ri bo‘ladigan qilib qismlarga ajratish kerak. U holda ikki o‘lchovli integralning 3) xossasiga ko‘ra ikki o‘lchovli integral bu qismlar bo‘yicha olingan ikki o‘chovli integrallar yig‘indisiga teng. 8.12-rasmda D integrallash sohasi Oy o‘q yo‘nalishi bo‘yicha to‘g‘ri bo‘lgan uchta sohaga ajratilgan.

Ikki o‘lchovli integralni hisoblash takroriy integralni hisoblashga keltirilganda, o‘zgaruvchilarning tartibini tanlash hisoblash murakkabligiga katta ta’sir qilishini ta’kidlab o‘tish kerak.

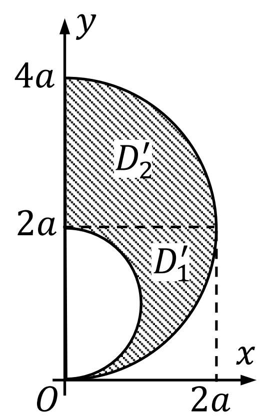
8.4-Misol. Dsoha 8.13 va 8.14-rasmlarda tasvirlangan bo‘lib, u $x=\sqrt{2ay-y^2}$ va $x=\sqrt{4ay-y^2}$ tenglamalar bilan berilgan aylanalar bilan chegaralangan. Dsohada integrallanuvchi $f(x,y)$ funksiyaning ikki o‘lchovli integralini takroriy integrallar orqali ifodalang.



8.12-rasm



8.13-rasm



8.14-rasm

► Ikki o‘lchovli integralni (8.17) formula bilan hisoblash uchun D sohani $x = a$ to‘g‘ri chiziq yordamida uchta qismga ajratamiz:

$D_1 = \left\{(x, y) \in \mathbb{R}^2, x \in [0, a], 2a - \sqrt{4a^2 - x^2} \le y \le a - -\sqrt{a^2 - x^2}\right\}$;

$D_2 = \left\{(x, y) \in \mathbb{R}^2, x \in [0, a], a + \sqrt{a^2 - x^2} \le y \le 2a + +\sqrt{4a^2 - x^2}\right\}$;

$D_3 = \left\{(x, y) \in \mathbb{R}^2, x \in [a, 2a], 2a - \sqrt{4a^2 - x^2} \le y \le 2a + +\sqrt{4a^2 - x^2}\right\}$.

U holda $f(x, y)$ funksiyaning D soha bo‘yicha ikki o‘lchovli integrali uchta takroriy integrallar yig‘indisi ko‘rinishida tasvirlanadi:

$$\iint_D f(x, y)dxdy = \int_0^a dx \int_{2a-\sqrt{4a^2-x^2}}^{a-\sqrt{a^2-x^2}} f(x, y)dy +$$
$$+\int_0^a dx \int_{a+\sqrt{a^2-x^2}}^{2a+\sqrt{4a^2-x^2}} f(x, y)dy + \int_a^{2a} dx \int_{2a-\sqrt{4a^2-x^2}}^{2a+\sqrt{4a^2-x^2}} f(x, y)dy.$$

(8.20) formulani qo‘llash uchun D sohani $y = 2a$ to‘g‘ri chiziq bilan ikkita sohaga ajratish yetarli (8.14-rasm).

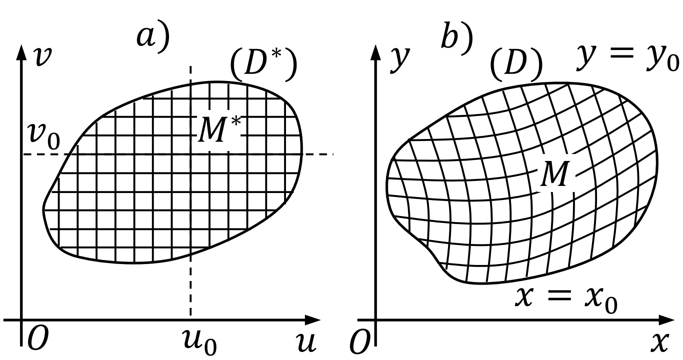
$D_1' = \left\{(x, y) \in \boldsymbol{R}^2, y \in [0,2a], \sqrt{2ay - y^2} \le x \le \sqrt{4ay - y^2}\right\}$;

$D_2' = \left\{(x, y) \in \boldsymbol{R}^2, y \in [2a, 4a], 0 \le x \le \sqrt{4ay - y^2}\right\}$.

Bu holda

$$\iint_D f(x, y)dxdy = \int_0^{2a} dy \int_{\sqrt{2ay-y^2}}^{\sqrt{4ay-y^2}} f(x, y)dx + \int_{2a}^{4a} dy \int_0^{\sqrt{4ay-y^2}} f(x, y)dx. ◄$$

8.5. Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish



8.15-rasm

Nuqtaning egri chiziqli koordinatalari. Ouv tekislikning D^* sohasida uzluksiz va uzluksiz xususiy hosilalarga ega bo‘lgan

$$\begin{cases} x = x(u, v), \\ y = y(u, v) \end{cases} \qquad (8.21)$$

funksiyalar juftligi aniqlangan bo‘lsin. (8.21) tenglamalar orqali D^* sohaning har bir $M^*(u, v)$ nuqtasiga Oxy tekislikning aniq bir $M(x, y)$ nuqtasi mos keladi va demak D^* sohaga Oxy tekislikning biror D sohasi mos kelar ekan (8.15-rasm). Bu holda (8.21) funksiyalar D^* sohani D sohaga *akslantiradi* va D soha D^* sohaning (8.21) akslantirishdagi *obrazi* deyiladi.

Har xil (u, v) nuqtalarga har xil (x, y) nuqtalar mos keladi deb faraz qilamiz. Bu esa (8.21) tenglamalar u va v o‘zgaruvchilarga nisbatan:

$$\begin{cases} u = u(x, y), \\ v = v(x, y) \end{cases} \tag{8.22}$$

bir qiymqtli yechilishiga teng kuchli. Bu holda akslantirish D^* sohani D sohaga *o‘zaro bir qiymatli akslantiradi* deb ataladi. Bunday akslantirishda D^* sohada yotuvchi ixtiyoriy L^* uzluksiz egri chiziq D sohada yotuvchi L uzluksiz egri chiziqqa o‘tadi. Agar $u(x, y)$, $v(x, y)$ funksiyalar ham uzluksiz bo‘lsa, u holda ixtiyoriy $L \in D$ uzluksiz egri chiziq (8.22) akslantirish yordamizda uzluksiz $L^* \in D^*$ egri chiziqqa o‘tadi.

u, v o‘zgaruvchilarning D^* sohadan olingan u_0, v_0 juftligi nafaqat $M^*(u_0, v_0) \in D^*$ nuqtaning o‘rnini, balki unga mos $M(x_0, y_0) \in D, x_0 = x(u_0, v_0), y_0 = y(u_0, v_0)$ nuqtaning ham o‘rnini aniqlaydi. Bu esa u, v sonlarni Oxy tekislik D sohasi M nuqtasining yangi koordinatalari deb qarashga asos bo‘ladi. Ular M nuqtaning *egri chiziqli koordinatalari* deb ataladi.

D sohaning bitta koordinatasi o‘garmas, ikkinchisi esa o‘zgaruvchan bo‘ladigan nuqtalari to‘plamiga *koordinata chizig‘i* deb ataladi. (8.21) tengliklarda $u = u_0$ deb olib, koordinata chizig‘ining

$$x = x(u, v_0), y = y(u, v_0) \tag{8.23}$$

parametrik tenglamasini hosil qilamiz. Bu yerda parametr rolini u o‘zgaruvchi bajaradi.v koordinataga mumkin bo‘lgan turli qiymatlarni berib, koordinata ciziqlaring Oxy tekislikdagi $(v = \text{const})$ *oilasini* hosil qilamiz. Xuddi shu singari boshqa $(u = \text{const})$ koordinata chiziqlari oilasini hosil qilish mumkin.

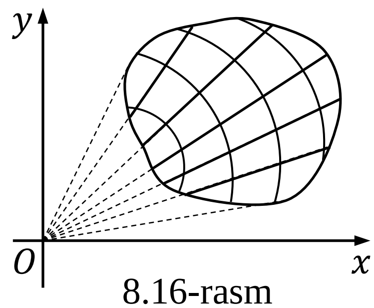
Egri chiziqli koordinatalarga misol sifatida qutb koordinatalarini olish mumkin. Bu holda (8.21) formulalar

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi \end{cases} \tag{8.24}$$

ko‘rinishda bo‘ladi.

Agar koordinatalar boshi D sohada yotmasa va $-\pi/2 \leq \varphi \leq \pi/2$ bo‘lsa, (8.24) formulalar r va φ o‘zgaruvchilarga nisbatan quyidagicha bir qiymatli yechiladi:

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \varphi = \operatorname{arctg}\dfrac{y}{x}. \end{cases}$$



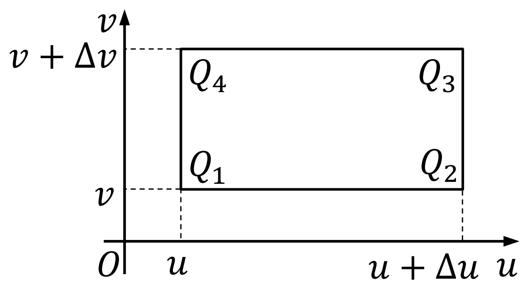
8.16-rasm

Shunday qilib, ko‘rsatilgan shartlarda (x, y) va (r, φ) juftliklar orasida o‘zaro bir qiymatli moslik mavjud bo‘ladi. r va φ larning koordinata chizig‘i $x^2 + y^2 = const$ aylanalar va koordinatalar boshidan chiquvchi nurlardan iborat bo‘ladi (8.16-rasm).

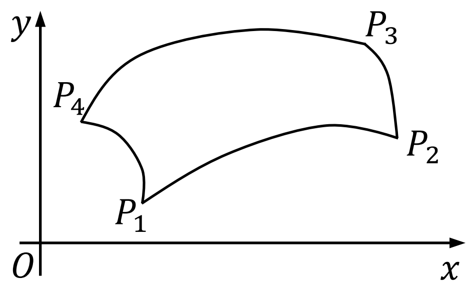
Egri chiziqli koordinatalarda yuza. Ouv tekislikda uchlari $Q_1(u, v)$, $Q_2(u + \Delta u, v)$, $Q_3(u + \Delta u, v + \Delta v)$, $Q_3(u, v + \Delta v)$ nuqtalarda bo‘lgan to‘g‘ri to‘rburchakni olamiz (8.17-rasm). Uning Oxy tekislikdagi tasviri ikki juft koordinata chiziqlaridan hosil bo‘lgan egri chiziqli to‘rtburchak bo‘ladi (8.18-rasm). Uning uchlari

$$P_1(x(u, v), y(u, v)), P_2(x(u + \Delta u, v), y(u + \Delta u, v)),$$
$$P_3(x(u + \Delta u, v + \Delta v), y(u + \Delta u, v + \Delta v))$$
$$\text{va } P_4(x(u, v + \Delta v), y(u, v + \Delta v))$$

nuqtalarda bo‘ladi.



8.17-rasm



8.18-rasm

$\delta = \sqrt{(\Delta u)^2 + (\Delta v)^2}$ deb qisqa belgilash kiritamiz. U holda $x(u, v)$ va $y(u, v)$ funksiyalar D^* sohada $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}$ va $\dfrac{\partial y}{\partial v}$ uzluksiz xususiy hosilalarga ega deb faraz qilsak, P_1 , P_2 , P_3 va P_4 nuqtalarni $o(\delta)$ aniqlikda $P_1'(x(u, v), y(u, v))$,

$$P_2'\left(x(u, v) + \frac{\partial x}{\partial u}\Delta u, y(u, v) + \frac{\partial y}{\partial u}\Delta u\right),$$
$$P_3'\left(x(u, v) + \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v, y(u, v) + \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v\right),$$
$$P_4'\left(x(u, v) + \frac{\partial x}{\partial v}\Delta v, y(u, v) + \frac{\partial y}{\partial v}\Delta v\right)$$

nuqtalar bilan almashtirish mumkin. U holda$\overrightarrow{P_1'P_2'} = \left\{\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\right\}va\overrightarrow{P_4'P_3'} = \left\{\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\right\}$bo‘lganligi uchun, $\overrightarrow{P_1'P_2'} = \overrightarrow{P_4'P_3'}$ bo‘ladi, bundan esa $P_1'P_2'P_3'P_4'$ figuraning parallelogramm ekanligi kelib chiqadi. Uning yuzi esa $\overrightarrow{P_1'P_2'}$ va $\overrightarrow{P_2'P_3'}$vektorlar vektor ko‘paytmasining absolyut qiymatiga teng:$S_{P_1'P_2'P_3'P_4'} = \left|\overrightarrow{P_1'P_2'} \times \overrightarrow{P_2'P_3'}\right|$. Bundan tashqari $\overrightarrow{P_2'P_3'} = \left\{\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v\right\}$, demak

$$\overrightarrow{P_1'P_2'} \times \overrightarrow{P_2'P_3'} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{vmatrix} = \vec{k}\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}\Delta u\Delta v\vec{k}.$$

Tenglikning o‘ng tomonidagi

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad (8.25)$$

determinant $x(u, v)$ va $y(u, v)$ funksiyalarning *Yakobi determinant* yoki *yakobiani* deb ataladi.

Shunday qilib,

$$\overrightarrow{P_1'P_2'} \times \overrightarrow{P_2'P_3'} = J(u, v)\Delta u\Delta v\vec{k}$$

tenglikka ega bo‘ldik, bundan esa parallelogrammning yuzi uchun

$$S_{P_1'P_2'P_3'P_4'} = |J(u, v)|\Delta u\Delta v$$

tenglikni hosil qilamiz.

Bundan esa $P_1P_2P_3P_4$ egri chiziqli to‘rtburchakning yuzi ham $o(\delta^2)$ aniqlikda $|J(u, v)|\Delta u\Delta v$ ifodaga teng ekanligi kelib chiqadi. Shuning uchun egri chiziqli koordinatalarda yuz elementi sifatida $|J(u, v)|dudv$ miqdor olinadi.

Xususiy holda, qutb koordinatalarida

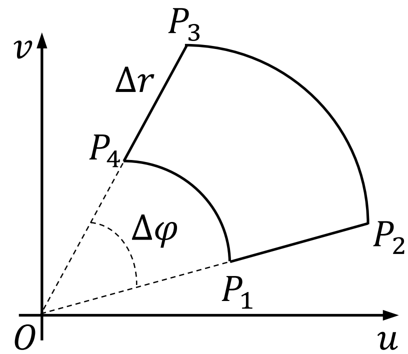
$$x(r, \varphi) = r\cos\varphi\,, y(r, \varphi) = r\sin\varphi$$

bo‘ladi va demak

$$J(r,\varphi)=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r}\\ \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & \sin\varphi\\ -r\sin\varphi & r\cos\varphi\end{vmatrix}=r,$$

har doim $r\geq 0$, u holda qutb koordinatalarida yuza elementi $ds=rdrd\varphi$ tenglikni qanoatlantiradi.

Shuning uchun $P_1P_2P_3P_4$ figurani yuqori tartibli cheksiz kichik miqdorlar aniqligida tomonlari $P_1P_2=\Delta r$ va $P_1P_4=r\Delta\varphi$ bo‘lgan to‘g‘ri to‘rtburchak deb qarash mumkin (8.19-rasm).



8.19-rasm

Egri chiziqli koordinatalarda yuzalarni hisoblash. Ouv tekislikning D^* sohasi Oxy tekislik D sohasining (8.22) akslantirishdagi obrazi bo‘lsin. D^* sohani $u=const$ va $v=const$ to‘g‘ri chiziqlar bilan elementar to‘g‘ri to‘rtburchaklarga ajratamiz. U holda D soha koordinata chiziqlari bilan elementar to‘rtburchaklarga bo‘linadi. $ik-$ to‘rtburchakning yuzi $|J(u_i,v_k)|\Delta u_i\Delta v_k+o(\delta_{ik}^2)$ ga teng bo‘ladi. Bu to‘rtburchaklarni yig‘ib, D sohaning yuzi taqriban

$$S_D\approx\sum_{i,k}|J(u_i,v_k)|\Delta u_i\Delta v_k$$

qiymatga teng ekanligini hosil qilamiz. D^* soha to‘g‘ri to‘rtburchakalri diametrlarining eng kattasini d orqali belgilab, uni nolga intiltiramiz. U holda $x(u,v)$ va $y(u,v)$ funksiyalarning uzluksizligi tufayli Oxy tekislikdagi elementar to‘rtburchaklarning diametrlari ham nolga intiladi. $J(u,v)$yakobianning ham D^*sohadagi uzluksizligini inobatga olsak, yuqorida aytilgan limitga o‘tsak D sohaning yuzi uchun

$$S_D=\iint_{D^*}|J(u,v)|dudv \tag{8.26}$$

aniq tenglikni hosil qilamiz.

Bu formula xususan yakobianning geometrik ma’nosini ochishga imkon beradi. Buning uchun (8.26) formulaga o‘rta qiymat haqidagi teoremaning (8.12) formulasini qo‘llaymiz:

$$S_D=|J(\xi,\eta)|S_{D^*}, \tag{8.27}$$

bundan esa

$$\frac{S_D}{S_{D^*}} = \left|J(\xi,\eta)\right|, \tag{8.28}$$

bu yerda (ξ,η) −nuqta D^* sohaning biror ichki nuqtasi.

Faraz qilaylik, D^* soha kichik va(u,v) −uning ixtiyoriy nuqtasi bo‘lsin. U holda (8.28) formulaga ko‘ra

$$\frac{S_D}{S_{D^*}} \approx |J(u,v)|,$$

yoki

$$S_D \approx |J(u,v)|S_{D^*}.$$

Bu so‘ngi tenglikdan quyidagi xulosani chiqarish mumkin: Ouv tekislikni Oxy tekislikka (8.21) akslantirishda $|J(u,v)|$ miqdor Ouv tekislikni (u,v) nuqtadagi cho‘zilish koeffisiyentini berar ekan.

Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish. $f(x,y)$ funksiya D sohada integrallanuvchi va $\iint\limits_D f(x,y)dxdy = I$ bo‘lsin, hamda D soha (8.22) akslantirish yordamida Ouv tekislikning D^* sohasiga akslantirilsin. Bu akslantirishni o‘zaro bir qiymatli deb faraz qilamiz. D^* sohani yuzlari $\Delta\sigma_1$, $\Delta\sigma_2$, …, $\Delta\sigma_n$ bo‘lgan ΔD_1^*, ΔD_2^*, …, ΔD_n^* sohalarga bo‘lamiz. U holda D soha ham yuzlari ΔS_1, ΔS_2, …, ΔS_n bo‘lgan ΔD_1, ΔD_2, …, ΔD_n sohalarga bo‘linadi. Har bir ΔD_i sohada ixtiyoriy ravishda bittadan nuqta tanlaymiz va

$$\tilde{S} = \sum_{i=1}^{n} f(x_i,y_i)\Delta S_i$$

yig‘indini tuzamiz. (8.27) formulaga asosan

$$\Delta S_i = |J(\xi_i,\eta_i)|\Delta\sigma_i,$$

bu yerda (ξ_i,η_i) nuqta ΔD_i^* sohaning biror nuqtasi. (x_i,y_i) nuqta sifatida aynan (ξ_i,η_i) nuqtaga akslanadigan ΔD_i sohaning nuqtasini olamiz (I integral mavjud, shu tufayli integral yig‘indida (x_i,y_i) nuqtani ixtiyoriy ravishda tanlash mumkin). Boshqacha qilib aytganda

$$x_i = x(\xi_i,\eta_i),\ \ y_i = y(\xi_i,\eta_i)$$

deb olamiz. U holda integral yig‘indi

$$\tilde{S} = \sum_{i=1}^{n} f(x(\xi_i,\eta_i), y(\xi_i,\eta_i))|J(\xi_i,\eta_i)|\Delta\sigma_i$$

ko‘rinishni oladi. $f(x,y)$ funksiyani uzluksiz deb faraz qilib va ΔD_k^* sohalar diametrlarining eng kattasini nolga intiltirsak

$$I=\iint\limits_{D^*} f(x(u,v),y(u,v)|J(u,v)|d\sigma$$

ya’ni

$$\iint\limits_{D} f(x,y)dxdy=\iint\limits_{D^*} f(x(u,v),y(u,v)|J(u,v)|dudv \qquad (8.29)$$

tenglikka ega bo‘lamiz.

Bu formula *ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish* formulasi deb ataladi. (8.26) formula (8.29) formulaning $f(x,y)=1$ bo‘lganda xususiy holi ekanligi ravshan.

Xususiy holda, qutb koordinatalarida (8.29) formula

$$\iint\limits_{D} f(x,y)dxdy=\iint\limits_{D^*} f(r\cos\varphi, r\sin\varphi)rdrd\varphi \qquad (8.30)$$

ko‘rinishda yoziladi.

Qutb koordinatalarida ikki o‘lchovli integralni hisoblash ham uni takroriy integralga keltirish orqali amalga oshiriladi. Masalan, agar D^* soha 8.20-rasmda tasvirlangan ko‘rinishda bo‘lsa ($\varphi=\alpha$ va $\varphi=\beta$ nurlar $r=r_1(\varphi)$ va $r=r_2(\varphi)$ chiziqlar bilan chegaralangan, ya’ni D^* soha to‘g‘ri soha: qutbdan chiqqan l nur uning chegaralarini faqat ikki nuqtada kesib o‘tmoqda), u holda (8.30) formulaning o‘ng tomonini quyidagicha yozish mumkin:

$$\iint\limits_{D^*} f(r\cos\varphi, r\sin\varphi)rdrd\varphi=\int\limits_{\alpha}^{\beta} d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} r\cdot f(r\cos\varphi, r\sin\varphi)dr. \qquad (8.31)$$

Eslatma. Faraz qilaylik ayrim nuqtalarda yoki hattoki ayrim chiziqlar bo‘ylab (8.29) formula o‘rinli bo‘ladigan shartlar, ya’ni yoki D va D^* sohalar o‘rtasidagi akslantirishning bir qiymatliligi, yoki $x(u,v)$ va $y(u,v)$ funksiyalarning, yoki ularning $\frac{\partial x}{\partial u},\frac{\partial x}{\partial v},\frac{\partial y}{\partial u}$ va $\frac{\partial y}{\partial v}$ xusussiy hosilalarining uzluksizligi bajarilmasin. Oxytekislikning bu nuqtalarini va chiziqlarini o‘zida saqlovchi biror kichik D_ε sohani olamiz va D_ε^* soha D_ε sohaning Ouv tekislikdagi obrazi bo‘lsin. $S_\varepsilon$vaσ_ε orqali mos ravishda D_ε va D_ε^* sohalarning yuzlarini belgilaymiz. U holda (8.29) formulaga ko‘ra

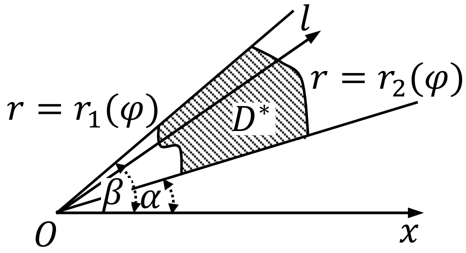
$$\iint\limits_{D\setminus D_\varepsilon} f(x,y)dxdy=\iint\limits_{D^*\setminus D_\varepsilon^*} f(x(u,v),y(u,v)|J(u,v)|dudv \qquad (8.32)$$

tenglikni yozish mumkin.

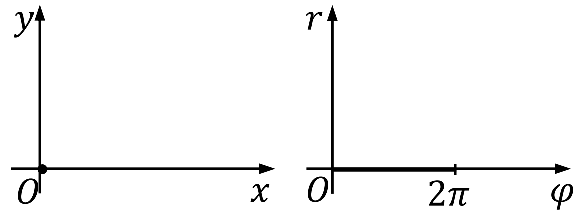
(8.32) tenglikning o‘ng tomonidagi integral umuman olganda

$\iint\limits_{D^*} f(x(u,v),y(u,v)|J(u,v)|dudv$ integraldan

$\iint\limits_{D_\varepsilon^*} f(x(u,v),y(u,v)|J(u,v)|dudv$ miqdorga farq qiladi.



8.20-rasm

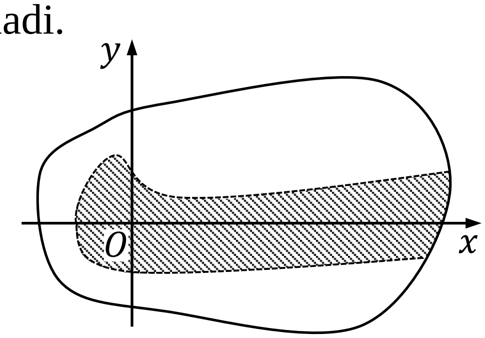


8.21-rasm

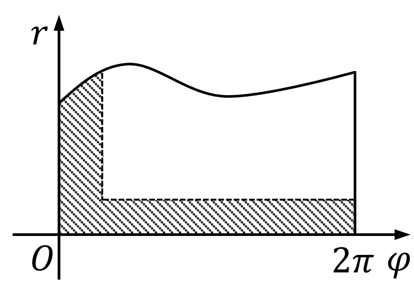
Xususiy holda, Oxy tekislikning $O(0,0)$ nuqtasiga qutb koordinatalar tekisligining $[0,2\pi]$ kesmada yotuvchi barcha (r,φ) nuqtalari mos keladi (8.21-rasm), ya'ni Oxy tekislikning $O(0,0)$ nuqtasiga akslantirishning o‘zaro bir qiymatlilik sharti bajarilmaydi. Shuning uchun agar koordinatalar boshi D sohada yotsa, (8.30) formula o‘rinli bo‘lmasligi ham mumkin. Biroq u 8.22 va 8.23-rasmlarda tasvirlangan bo‘yalmagan $D\backslash D_\varepsilon$ va $D^*\backslash D_\varepsilon^*$ sohalarda o‘rinli.

Agar $f(r\cos\varphi, r\sin\varphi)r$ miqdor D^* sohada chegaralangan bo‘lsa, u holda D_ε va D_ε^* sohalarning yuzini nolga intiltirib, bu hol uchun ham (8.30) formulani hosil qilamiz.

Mulohaza. Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish (aniq integraldagi singari), uni hisoblash uchun qulay bo‘ladigan ko‘rinishga keltirish maqsadida qo‘llaniladi. Biroq ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish nafaqat integral osti funksiyani, balki integrallash sohasini ham soddaroq ko‘rinishga keltiradi. Integrallash sohasini soddaroq ko‘rinishga keltirish integral osti funksiyani soddalashtirishga nisbatan muhimroq ham. Buning natijasida ikki o‘lchovli integralda integrallash chegaralarini qo‘yish va takroriy integralda hisoblash yengillashadi.



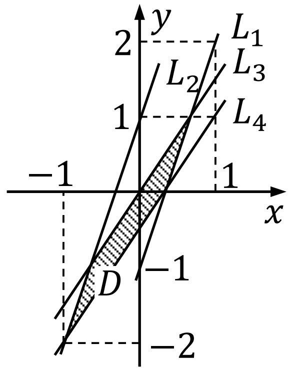
8.22-rasm



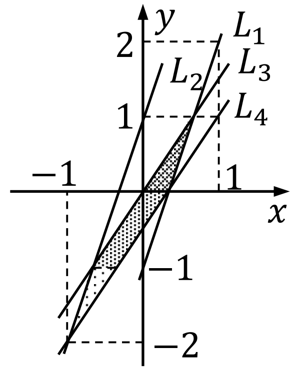
8.23-rasm

8.5-misol. $\iint\limits_D (2x-y)dxdy$ integralni hisoblang, bu yerda D −tomonlari $L_1: 3x-y-1=0$, $L_2: 3x-y+1=0$, $L_3: 3x-2y=0$, $L_4: 3x-2y-$

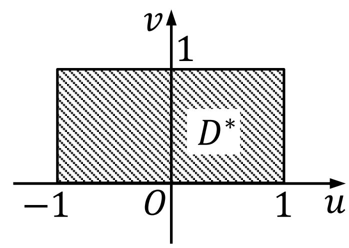
$1 = 0$ to‘g‘ri chiziqlarda yotuvchi parallelogrammning ichki qismi (8.24-rasm).



8.24-rasm



8.25-rasm



8.26-rasm

►Dekart koordinatalar sistemasida bu integralni (8.17) fo‘rmula bo‘yicha ham, (8.20) formula bo‘yicha ham takroriy integralga keltirib hisoblash uchun, integrallash sohasini uchgа bo‘lish kerak bo‘ladi (8.25-rasm). Agar $u = 3x - y$, $v = 3x - 2y$ formulalar orqali yangi o‘zgaruvchi kiritsak, mazkur integralni osongina hisoblash mumkin. Bunda $x(u, v) = (2u - v)/3$, $y(u, v) = u - v$ bo‘ladi va (x, y) nuqta paralleleogramm bo‘ylab harakatlanganda, yangi u va v o‘zgaruvchilar $-1 \leq u \leq 1, 0 \leq v \leq 1$ kesmalarda o‘zgaradi, ya’ni D^* $-$integrallash sohasi to‘g‘ri to‘rtburchakdan iborat bo‘ladi (8.26-rasm). Integral ostidagi funksiya $2x - y = (4u - 2v)/3 - (u - v) = (u + v)/3$ ko‘rinishni oladi.

Endi $x(u, v)$ va $y(u, v)$ funksiyalarning yakobianini topamiz. Buning uchun $x'_u = 2/3$, $x'_v = -1/3$, $y'_u = 1$, $y'_v = -1$ xususiy hosilalrni (8.25) formulaga qo‘yamiz:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{2}{3} & 1 \\ -\dfrac{1}{3} & -1 \end{vmatrix} = -\frac{1}{3}$$

U holda (8.29) formulaga ko‘ra qidirilayotgan integralni hisoblaymiz:

$$\iint_D (2x - y)dxdy = \iint_{D^*} \frac{u + v}{3} \left|-\frac{1}{3}\right| dudv = \frac{1}{3} \int_{-1}^{1} du \int_{0}^{1} \frac{u + v}{3} dv =$$

$$= \frac{1}{9} \int_{-1}^{1} \left[\left(uv + \frac{1}{2}v^2\right)\Big|_{-1}^{1}\right] du = \frac{1}{3} \int_{-1}^{1} \left(u + \frac{1}{2}\right) du = \frac{1}{9}\left(\frac{1}{2}u^2 + \frac{1}{2}u\right)\Big|_{-1}^{1} =$$

$$= \frac{1}{9}.$$

Yangi o‘zgaruvchilarga o‘tgandan so‘ng integrallash sohasi oddiy to‘g‘ri yuqorida ta’kidlaganimizga ko‘ra to‘g‘ri to‘rtburchakka aylandi.◄

Mulohaza. Ikki o‘lchovli integralda o‘zgaruvchilarni shunday almashtirish kerakki, natijada ham integral osti funksiya, ham integrallash sohasi soddaroq ko‘rinishni olsin. Quyida qutb koordianataliga o‘tishning o‘ziga xos xususiyatlarini keltiramiz:

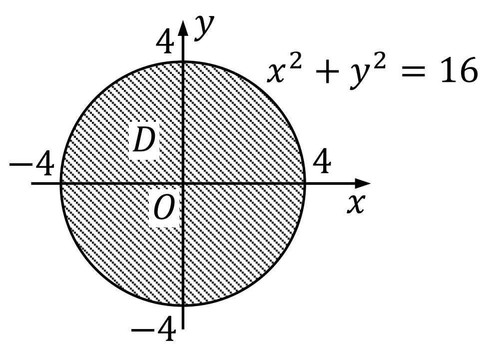
1) Integral osti funksiya $f(x^2+y^2)$ ko‘rinishda; D integrallash sohasi doira, xalqa yoki ularni qismi bo‘lganda qutb koordinatalariga o‘tish qulay.

2) Amalda qutb koordinatalariga $x=r\cos\varphi, y=r\sin\varphi, dxdy=rdrd\varphi$ ko‘rinishdagi almashtirish orqali o‘tiladi; D sohani chegaralovchi chiziq ham qutb koordinatalari orqali ifodalanadi. (x,y) nuqtaning D sohada o‘zgarish qonuniyatiga qarab yangi r,φ o‘zgaruvchilar uchun o‘zgarish chegaralari aniqlanadi.

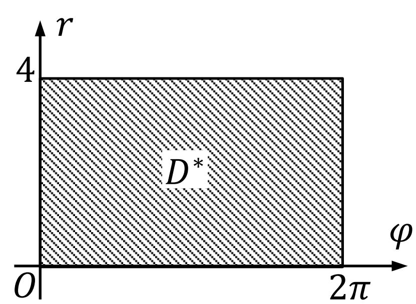
8.6-Misol. $\iint\limits_D \sqrt{16-x^2-y^2}dxdy$ integralni hisoblang, bu yerda D −integrallash sohasi $x^2+y^2\le 16$ doiradan iborat (8.27-rasm).

►(8.30) formulani qo‘llab, qutb koordinatalariga o‘tamiz:

$$\iint\limits_D \sqrt{16-x^2-y^2}dxdy=\iint\limits_{D^*} \sqrt{16-(r\cos\varphi)^2-(r\sin\varphi)^2}\cdot rdrd\varphi=$$
$$=\iint\limits_{D^*} r\sqrt{16-r^2}drd\varphi.$$



8.27-rasm



8.28-rasm

Qutb koordinatalar sistemasidagi D^* integrallash sohasi $0\le\varphi\le 2\pi, 0\le r\le 4$ tengsizlilar bilan aniqlanadi (8.28-rasm). Demak D doira qutb koordinatalariga o‘tganda D^* to‘g‘ri to‘rtburchakka o‘tar ekan. (8.31) formulaga ko‘ra

$$\iint\limits_{D^*} r\sqrt{16-r^2}drd\varphi=\int\limits_0^{2\pi} d\varphi\int\limits_0^{3} r\sqrt{16-r^2}dr=$$

$$= -\frac{1}{2}\int_0^{2\pi} d\varphi \int_0^3 (16-r^2)^{1/2} d(16-r^2) = \int_0^{2\pi} \left[\frac{(16-r^2)^{3/2}\cdot 2}{3}\Big|_0^3\right] d\varphi =$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(0-\frac{128}{3}\right) d\varphi = \frac{256\pi}{90},$$

qidirilgan ikki o‘lchovli integralning qiymatini hosil qildik.◄

8.7-Misol. $I = \iint_D \frac{1}{\sqrt{1+x^2+y^2}} dxdy$ integralni hisoblang, bu yerda D integrallash sohasi$D = \{(x,y): x^2+y^2 \le 1, x \ge 0, y \ge 0\}$ –birlik doiraning birinchi chorakda yotgan qismi.

► $x = r\cos\varphi$, $y = r\sin\varphi$ qutb koordinatalariga o‘tamiz. U holda integrallash sohasi

$$D^* = \{(r,\varphi): 0 \le r \le 1, 0 \le \varphi \le \pi/2\}$$

to‘g‘ri to‘rtburchakdan iborat bo‘ladi. O‘zgaruvchilari almashtirilgandan so‘ng integral osongina hisoblanadi:

$$I = \iint_{D^*} \frac{rdrd\varphi}{\sqrt{1+r^2}} = \int_0^{\pi/2} d\varphi \int_0^3 \frac{rdr}{\sqrt{1+r^2}} = \frac{\pi}{2}\sqrt{1+r^2}\Big|_0^1 = \frac{\pi(\sqrt{2}-1)}{2}. \quad ◄$$

Mulohaza. Agar D sohada yakobian noldan farqli bo‘lsa, bu sohaning har bir nuqtasining biror atrofidagi akslanish o‘zaro bir qiymatli bo‘ladi. Biroq shuning bilan birga butun sohaning akslanishi bir qiymatli bo‘lmasligi ham mumkin.

$$x = e^u\cos v, y = e^u \sin v, -\infty < u,v < +\infty$$

funksiyalar bilan aniqlanadigan akslanishni qaraymiz. Ularning yakobiani

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u\cos v & e^u \sin v \\ -e^u\sin v & e^u\cos v \end{vmatrix} = e^u$$

butun tekislikda noldan farqli. Shunga qaramasdan $u = 0, v = 0$ va $x = 0, v = 2\pi$ uchun $x = 1, y = 0$ bo‘ladi, ya’ni bu akslantirish o‘zaro bir qiymatli emas.

Boshqa tomondan, agar akslantirishning yakobiani biror nuqtada nolga aylangani bilan, bu nuqtaning atrofida akslanish o‘zaro bir qiymatli bo‘lishi mumkin. Masalan,

$$x = u^3, y = v^3, -\infty < u,v < +\infty$$

funksiyalar bilan aniqlanadigan akslanishning yakobiani $J = 9u^2v^2$ bo‘ladi va u $u = 0$ va $v = 0$ bo‘lsa nolga aylanadi, ammo bu akslanish o‘zaro bir qiymatli.

8.6. Ikki o‘lchovli integralning tadbiqlari

Silindrik jismning hajmi. Ikki o‘lchovli integral tushunchasiga keltiriladigan masala sifatida yuqoridan $z = f(x, y) \geq 0$ sirt bilan, quyidan Oxy tekislikning D sohasi bilan, yon tomonlardan esa yo‘naltiruvchisi D sohaning chegarasi bo‘lgan silindrik sirt bilan chegaralangan jism hajmini hisoblash qaralgan edi va bu hajm

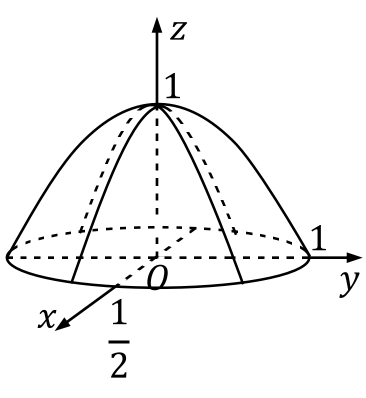
$$V = \iint_D f(x, y)dxdy \tag{8.33}$$

ikki o‘lchovli integral bilan hisoblanishini ko‘rgan edik.

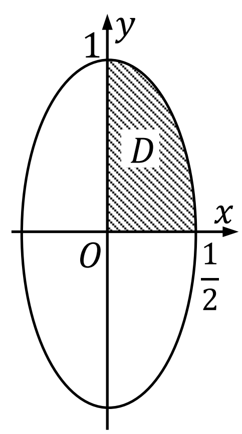
Shunday qilib, *nomanfiy funksiya ikki o‘lchovli integralining qiymati silindrik jismning hajmiga teng* bo‘lar ekan.

8.8-Misol. $z = 1 - 4x^2 - y^2$ sirt va Oxy tekislik bilan chegaralangan jismning hajmini hisoblang (8.29-rasm).

► $z = 1 - 4x^2 - y^2$ elliptik paraboloid Oxy tekislik bilan L: $\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{1^2} = 1$, $z = 0$ chiziq bo‘ylab kesishadi. Bu chiziq yarim o‘qlari $a = 1/2$ va $b = 1$ bo‘lgan ellipsdan iborat (8.30-rasm) va demak integrallash sohasi ana shu ellipsning ichki qismi bo‘ladi.



8.29-rasm



8.30-rasm

Qaralayotgan jismning Oxz va Oyz tekisliklarga nisbatan simmetrikligi tufayli, integrallsh sohasini ellipsning $1/4$ qismini olib, natijani to‘rtga ko‘pqytirish kerak:

$$V = 4\iint\limits_D (1-4x^2-y^2)dxdy = 4\int\limits_a^{1/2} dx \int\limits_0^{\sqrt{1-4x^2}} (1-4x^2-y^2)dy =$$

$$= 4\int\limits_a^{1/2} \left[(1-4x^2)y - \frac{1}{3}y^3\right]_0^{\sqrt{1-4x^2}} dx = \frac{8}{3}\int\limits_a^{1/2} (1-4x^2)^{3/2}dx =$$

$$= \left|\begin{array}{l} 2x = \sin t, \qquad 2dx = \cos t\, dt \\ 1-4x^2 = 1-\sin^2 t = \cos^2 t \\ x_1 = 0, t_1 = 0, x_2 = \dfrac{1}{2}, t_2 = \dfrac{\pi}{2} \end{array}\right| = \frac{4}{3}\int\limits_0^{\pi/2} (\cos^2 t)^{3/2}\cos t\, dt =$$

$$= \frac{4}{3}\int\limits_0^{\pi/2} \cos^4 t\, dt = \frac{4}{3}\int\limits_0^{\pi/2} \frac{1+2\cos 2t+\cos^2 2t}{4}dt =$$

$$= \frac{1}{3}(t+\sin 2t)\Big|_0^{\pi/2} + \frac{1}{3}\int\limits_0^{\pi/2} \frac{1+\cos 4t}{2}dt =$$

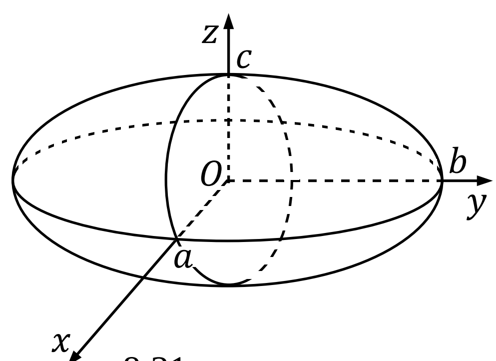
$$= \frac{\pi}{6} + \frac{1}{6}\left(t + \frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2} = \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}. \quad ◀$$

8.9-Misol. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ellipsoid bilan chegaralangan jismning hajmini hisoblang (8.31-rasm).

▶ Qaralayotgan jismning Oxy tekislikka nisbatan simmetrikligi tufayli uning hajmi

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

funksiyaning Oxy tekislikdagi $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



8.31-rasm

ellips bilan chegaralangan D soha bo‘yicha olingan ikki o‘lchovli integralning ikkilanganligiga teng:

$$V = 2\iint\limits_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dxdy.$$

Bu integralni hisoblash uchun $x = ar\cos\varphi$, $y = ar\sin\varphi$ deb o‘zgaruvchilarni almashtiramiz. U holda integral ostidagi funksiya $z = c\sqrt{1-r^2}$ ko‘rinishga o‘tadi, D integrallash sohasiga $D^* = \{(r,\varphi): 0 \le r \le$

$1,\ 0 \le \varphi \le 2\pi\}$ to‘g‘ri to‘rtburchak mos keladi. $x'_r = a\cos\varphi$, $x'_\varphi = -ar\sin\varphi$, $y'_r = b\sin\varphi$, $y'_\varphi = br\cos\varphi$ xususiy hosilalarni (8.25) formulaga qo‘yib, bu almashtirishning yakobianini hisoblaymiz:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\varphi & b\sin\varphi \\ -ar\sin\varphi & br\cos\varphi \end{vmatrix} = abr.$$

Shuning uchun (8.29) formulaga ko‘ra

$$V = 2\iint_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dxdy = 2\iint_D c\sqrt{1-r^2}\,abrdrd\varphi =$$

$$= 2abc\int_0^{2\pi} d\varphi \int_0^1 r\sqrt{1-r^2}dr = 2abc\int_0^{2\pi} d\varphi \cdot \int_0^1 r\sqrt{1-r^2}dr =$$

$$= -\frac{4\pi}{3}abc(1-r^2)^{\frac{3}{2}}\Big|_0^1 = \frac{4}{3}\pi abc. \blacktriangleleft$$

Yassi figuraning yuzi. Ikki o‘lchovli integralning xossasiga ko‘ra Oxy tekislik *D sohasining S yuzi $f(x,y) \equiv 1$ funksiyaning bu soha bo‘yicha olingan ikki o‘lchovli integraliga teng*:

$$S = \iint_D dxdy. \qquad (8.34)$$

8.10-Misol. $(x^2+y^2)^2 = 8xy$ chiziq bilan chegaralangan figuraning yuzini hisoblang (8.32-rasm).

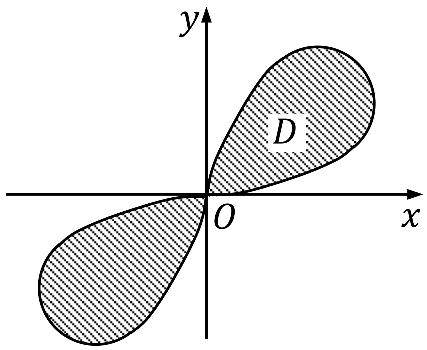
►Bu yerda $x = r\cos\varphi$, $y = r\sin\varphi$ tengliklar bilan qutb koordinatalarini kiritamiz. Egri chiziq tenglamasi $r^4 = 8r^2\cos\varphi\sin\varphi$ ko‘rinishni oladi, bundan esa $r = 2\sqrt{\sin 2\varphi}$ tenglikka ega bo‘lamiz. Egri chiziq va demak u chegaralagan soha koordinatalar boshiga nisbatan simmetrik. Shuning uchun

$$S = \iint_D dxdy = 2\int_0^{\pi/2} d\varphi \int_0^{2\sqrt{\sin 2\varphi}} rdr = \int_0^{\pi/2} \left(r^2\Big|_0^{2\sqrt{\sin 2\varphi}}\right)d\varphi =$$

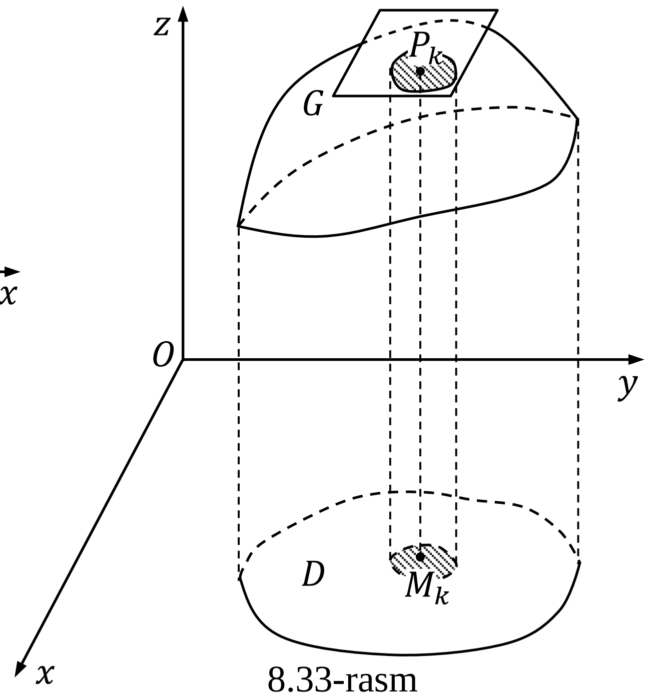
$$= \int_0^{\pi/2} 4\sin 2\varphi\, d\varphi = -2\cos 2\varphi\Big|_0^{\pi/2} = 4. \blacktriangleleft$$

Sirt yuzi. G sirt $z = f(x,y)$ sirtning Oxy tekislikdagi $\overline{D}$ yopiq sohaga proyeksiyalanuvchi qismi bo‘lsin. Sirtning mazkur qismini silliq deb faraz qilamiz, ya’ni $f(x,y)$ funksiya $\overline{D}$ sohada uzluksiz va uzluksiz $f'_x(x,y)$,

$f'_y(x, y)$ xususiy hosilalarga ega. $\overline{D}$ sohani yuzalari ΔS_1,ΔS_2,…, ΔS_n bo‘lgan n ta ΔD_1 ,ΔD_2 ,…, ΔD_n qismlarga ajratamiz. Har bir ΔD_k sohada ixtiyoriy ravishda bittadan $M_k(x_k, y_k)$ nuqta tanlaymiz (8.33-rasm). Bu nuqtaga sirtning $P_k(x_k, y_k, f(x_k, y_k))$ nuqtasi mos keladi.



8.32-rasm



8.33-rasm

P_k nuqtada G sirtga urinma tekislik o‘tkazamiz. ΔG_kbu tekislikning ΔD_k sohaga proyeksiyalanuvchi qismi va $\Delta\sigma_k$ bu maydonchaning yuzi bo‘lsin. Bu yuzalardan

$$\tilde{\sigma} = \sum_{k=1}^{n} \Delta\sigma_k$$

yig‘indini tuzamiz.

Har doimgidek λ orqali ΔD_k sohalar diametrlarining eng kattasini belgilaymiz. Agar bo‘linishlar soni cheksiz oshganda va λ nolga intilganda chekli

$$\sigma = \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{k=1}^{n} \Delta\sigma_k \qquad (8.35)$$

limit mavjud bo‘lsa va bu limit $\overline{D}$ sohani bo‘lish usuliga, hamda M_k nuqtalarning tanlanishiga bog‘liq bo‘lmasa, u G sirtning yuzi deb ataladi.

G sirtga yuqorida qo‘yilgan shartlar bajarilganda, bu limitning mavjudligini isbotlaymiz.

G sirtning P_k nuqtasida o‘tkazilgan urinma tenglamasi (7.14) formulaga ko‘ra

$$z - z_k = \frac{\partial f(x_k, y_k)}{\partial x}(x - x_k) + \frac{\partial f(x_k, y_k)}{\partial y}(y - y_k)$$

ko‘rinishda bo‘ladi, bu yerda $z_k = f(x_k, y_k)$. U holda bu tekislikning Oxy tekislikka nisbatan γ_k og‘ma burchagi

$$\cos\gamma_k = \frac{1}{\sqrt{1 + \left(f'_x(x_k, y_k)\right)^2 + \left(f'_y(x_k, y_k)\right)^2}} \qquad (8.36)$$

tenglik bilan aniqlanadi. ΔS_kva$\Delta\sigma_k$ yuzalar o‘zaro $\Delta S_k = (\cos\gamma_k)\Delta\sigma_k$ tenglik bilan bog‘langan. (8.36) tenglikni inobatga olgan holda so‘ngi tenglikni (8.35) tenglikka qo‘ysak

$$\sigma = \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{k=1}^{n} \sqrt{1 + \left(f'_x(x_k, y_k)\right)^2 + \left(f'_y(x_k, y_k)\right)^2}\, \Delta S_k \qquad (8.37)$$

tenglik hosil bo‘ladi. Limit ostidagi yig‘indi

$$\sqrt{1 + \left(f'_x(x, y)\right)^2 + \left(f'_y(x, y)\right)^2}$$

funksiyaning $\bar{D}$ soha bo‘yicha integral yig‘indisini aniqlaydi. Qo‘yilgan shartlarda bu funksiya $\bar{D}$ sohada uzluksiz bo‘ladi va demak 8.1-Teoremaga ko‘ra (8.37) limit mavjud bo‘ladi. U holda ikki o‘lchovli integralning ta’tifiga ko‘ra

$$\sigma = \iint_D \sqrt{1 + \left(f'_x(x, y)\right)^2 + \left(f'_y(x, y)\right)^2}\, dxdy \qquad (8.38)$$

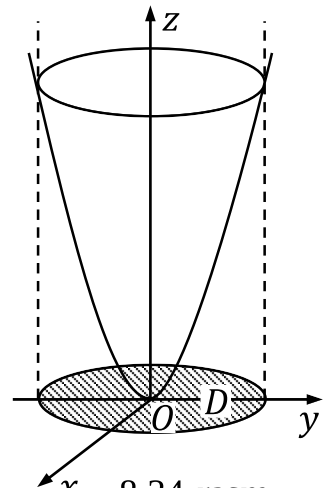
tenglikni hosil qilamiz. Bu formula *sirt yuzini hisoblash formulasi* deb ataladi.

8.11-Misol. $z = x^2 + y^2$ paraboloidning $x^2 + y^2 = 2$ silindr ichida joylashgan qismining yuzini toping (8.34-rasm).

► $z = x^2 + y^2$ funksiyaning $z'_x = 2x$ va $z'_y = 2y$ xususiy hosilalari $D: x^2 + y^2 \le 2$ doirada uzluksiz, shuning uchun (8.38) formulaga ko‘ra qaralayotgan sirtning σ yuzi

$$\sigma = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dxdy$$

ikki o‘lchovli integral bilan hisoblanadi. Integrallash sohasining shaklidan va integral osti funksiyaning ko‘rinishidan kelib chiqib $x = 2r\cos\varphi$ va $y = 2r\sin\varphi$ deb qutb koordinatalariga o‘tamiz:



8.34-rasm

$$\sigma = \iint_{D^*} \sqrt{1+4r^2}\, r dr d\varphi = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r\sqrt{1+4r^2}\, dr =$$

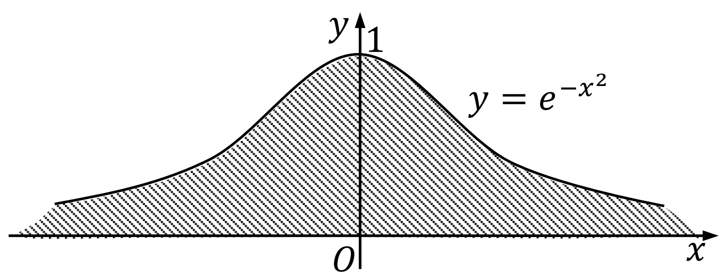
$$= \int_0^{2\pi} d\varphi \cdot \int_0^{\sqrt{2}} r\sqrt{1+4r^2}\, dr =$$

$$= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3}(1+4r^2)^{\frac{3}{2}}\Big|_0^{\sqrt{2}} = \frac{\pi}{6}(27-1) = \frac{13\pi}{3}.$$

va qidirilayotgan yuzani hosil qilamiz.◀

Puasson integralini hisoblash. 5.2 da aniqmas integrallarni o‘rganishda shunday elementar funksiyalar mavjud bo‘lib, ularning aniqmas integralini elemetar funksiyalarning chekli sondagi arifmetik amallari va superpozitsiyalari orqali ifodalab bo‘lmasligini ko‘rgan edik. Bulardan biri- Puasson integrali deb ataluvchi $\int e^{-x^2} dx$ integral bo‘lib, uning $(-\infty, +\infty)$ cheksiz oraliqdagi aniq integrali ehtimollar nazariyasida juda ko‘p qo‘llaniladi. Shunday qilib, $\int_{-\infty}^{+\infty} e^{-x^2} dx$ integralni hisoblaymiz. Bu aniq integralni ham Puasson integrali deb ataymiz va uni geometrik jihatdan $y = e^{-x^2}$ egri chiziq va Ox o‘q orasida joylashgan cheksiz $(-\infty < x < +\infty)$ figuranig yuzi sifatida talqin qilish mumkin (8.35-rasm).

Yuqorida ta’kidlab o‘tganimizga ko‘ra $\int e^{-x^2} dx$ aniqmas integralni elementar funksiyalar orqali ifodalab bo‘lmaydi, shuning uchun



8.35-rasm

$$\int_{-\infty}^{+\infty} e^{-x^2} dx$$

aniq integralni hisoblashga Nyuton-Leybnis formulasini qo‘llab bo‘lmaydi, demak uni hisoblashga maxsus usuldan foydalanishga to‘g‘ri keladi.

Dastlab, Puasson integrali yaqinlashuvchi birinchi tur xosmas integral ekanligini ta’kidlab o‘tish kerak. Haqiqatdan ham, barcha $x > 0$ uchun $e^x > x$ tengsizlik (8.36-rasm) o‘rinli bo‘ladi, bundan esa barcha x uchun $e^{x^2} > x^2$

tengsizlikni yozish mumkin bo‘ladi. U holda barcha x uchun $e^{-x^2} < 1/x^2$ bo‘ladi. 6.10-Misolga ko‘ra $1/x^2$ funksiyaning $(0, +\infty)$ oraliq bo‘yicha xosmas integrali yaqinlashuvchi bo‘ladi, bundan esa 6.14-Teoremaga ko‘ra $\int\limits_0^{+\infty} e^{-x^2}dx$ va $\int\limits_{-\infty}^{0} e^{-x^2}dx$ xosmas integrallarning yaqinlashuvchiligi kelib chiqadi. Shuning uchun $\int\limits_{-\infty}^{+\infty} e^{-x^2}dx$ xosmas integral ham yaqinlashadi va u bu xosmas integrallar yig‘indisiga teng. e^{-x^2} funksiyaning juftligidan

$$\int\limits_{-\infty}^{+\infty} e^{-x^2}dx = 2\int\limits_0^{+\infty} e^{-x^2}dx$$

tenglik kelib chiqadi.

Puasson integralini hisoblash uchun

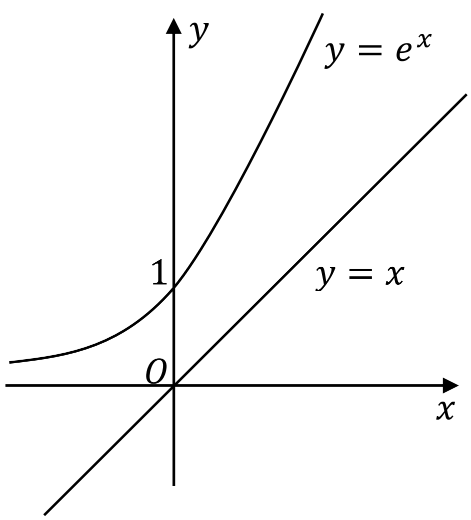
$$D_a = \{(x, y): -a \le x \le a, -a \le y \le a\}$$

kvadratni qaraymiz (8.37-rasm). Bu kvadarat bo‘yicha ikki o‘lchovli integralni hisoblaymiz:

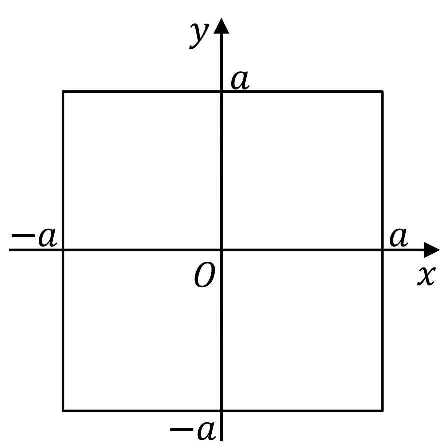
$$\iint\limits_{D_a} e^{-(x^2+y^2)}dxdy = \int\limits_{-a}^{a} dx \int\limits_{-a}^{a} e^{-x^2}e^{-y^2}dy = \int\limits_{-a}^{a} \left[e^{-x^2} \int\limits_{-a}^{a} e^{-y^2}dy\right] dx =$$

$= \int\limits_{-a}^{a} e^{-y^2}dy \cdot \int\limits_{-a}^{a} e^{-x^2}dx$, ya’ni

$$\iint\limits_{D_a} e^{-(x^2+y^2)}dxdy = \left(\int\limits_{-a}^{a} e^{-x^2}dx\right)^2 \tag{8.39}$$



8.36-rasm



8.37-rasm

Endi $a \to \infty$ bo‘lsin. U holda D_a integrallash sohasi bu limitda butun Oxy tekislikka aylanadi va uni biz D_∞ orqali belgilaymiz. Shuning uchun

$$\iint\limits_{D_\infty} e^{-(x^2+y^2)}dxdy = \left(\int\limits_{-\infty}^{\infty} e^{-x^2}dx\right)^2, \tag{8.40}$$

bundan esa Puasson integrali uchun

$$\int\limits_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\iint\limits_{D_\infty} e^{-(x^2+y^2)}dxdy} \tag{8.41}$$

tenglikni hosil qilamiz.

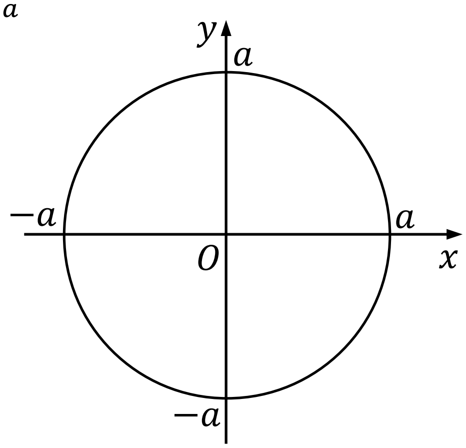
Shunday qilib, Puasson integralini hisoblash uchun (8.41) tenglikning o‘ng tomonidagi “xosmas” ikki o‘lchovli integralni hisoblash yetarli ekan. Integral ostidagi funksiyaning ko‘rinishidan qutb koordinatalariga o‘tish maqsadga muvofiq deb xulosaga kelish mumkin. Buning uchun 8.38-rasmda tasvirlangan $B_a = \{(x,y): x^2 + y^2 \leq a^2\}$ doirani kiritamiz. U holda

$$\iint\limits_{B_a} e^{-(x^2+y^2)}dxdy = \int\limits_{-\pi}^{\pi} d\varphi \int\limits_0^a e^{-r^2}rdr =$$

$$= 2\pi \int\limits_0^a re^{-r^2}dr = \pi \int\limits_0^a e^{-r^2}d(r^2) = \pi e^{-r^2}\Big|_a^0,$$

ya’ni,

$$\iint\limits_{B_a} e^{-(x^2+y^2)}dxdy = \pi\left(1 - e^{-a^2}\right). \tag{8.42}$$



8.38-rasm

Bu yerda $a \to \infty$ deb limitga o‘tsak,

$$\iint\limits_{B_\infty} e^{-(x^2+y^2)}dxdy = \pi \tag{8.43}$$

tenglikni hosil qilamiz. Biroq, (8.42) tenglikdan (8.43) tenglikga o‘tishda B_a integrallash sohasi cheksiz radiusli B_∞ doiraga, ya’ni butunOxytekislikga o‘tiladi. Shuning uchun B_∞ belgilashdan D_∞ belgilashga o‘tsak, (8.43) tenglikni

$$\iint\limits_{D_\infty} e^{-(x^2+y^2)}dxdy = \pi \tag{8.45}$$

ko‘rinishda yozish mumkin. U holda (8.41) tenglikka ko‘ra

$$\int\limits_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Puasson integralini hisoblash formulasini hosil qildik.

Yassi figura massasi. Chegaralangan yassi D figurada $\mu(P) = \mu(x,y) \geq 0$ sirt zichligiga ega bo‘lgan massa uzluksiz taqsimlangan bo‘lsin, bu yerda $\mu(x,y)$ funksiya D sohada uzluksiz. D sohani umumiy ichki nuqtalarga ega bo‘lmagan n ta

$$D_1,\ D_2, \ldots, D_n$$

qismlarga ajratamiz. Ularning yuzalarini mos ravishda

$$\Delta S,\ \Delta S_2, \ldots,\ \Delta S_n$$

orqali belgilaymiz. Har bir D_i $(i = 1,2, \ldots, n)$ qismda ixtiyoriy ravishda $P_i(x_i, y_i)$ nuqtani tanlaymiz, bu nuqtadagi zichlik $\mu(x_i, y_i)$ ga teng bo‘ladi. $\mu(x,y)$ funksiyaning uzluksizligi tufayli D figura D_i qismining m_i massasi taqriban $\mu(x_i, y_i)\Delta S_i$ ko‘paytmaga teng bo‘ladi, D figuraning m massasi mos ravishda

$$m \approx \sum_{i=1}^{n} \mu(x_i, y_i)\Delta S_i$$

yig‘indiga taqriban teng bo‘ladi. So‘ngi yig‘indi D sohada uzluksiz bo‘lgan $\mu(x,y)$ funksiyaning integral yig‘indisi bo‘ladi. Har doimgidek d orqali D_i qismiy sohalar diametrlarining eng kattasini belgilaymiz. So‘ngi taqribiy tenglikda d diametrni nolga intiltirsak,

$$m = \lim_{d\to 0} \sum_{i=1}^{n} \mu(x_i, y_i)\Delta S_i = \iint\limits_{D} \mu(x,y)dxdy \tag{8.46}$$

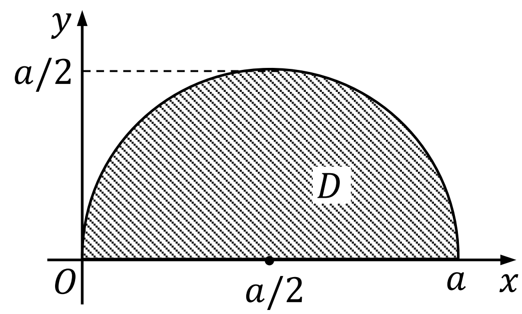
aniq tenglikni hosil qilamiz. Xususiy holda agar massa butun figura bo‘ylab tekis taqsimlangan, ya’ni $\mu = const$ bo‘lsa, (8.46) formula

$$m = \mu \iint\limits_{D} dxdy = \mu \cdot S$$

ko‘rinisni oladi, bu yerda S berilgan D sohaning yuzi.

8.12-Misol. Diametri a va markazi $C(a/2{,}0)$ nuqtada bo‘lgan yarim doiraviy figuraning m massasini hisoblang, massaning har bir nuqtadagi sirt zichligi bu nuqtadan koordinatalar boshigacha bo‘lgan masofaga proporsional (8.39-rasm).

►Demak, m massali D figura quyidan Ox bilan, yuqoridan esa $x^2+y^2=ax$ aylana bilan chegaralangan bo‘lib, uning sirt zichligi $\mu(x,y)=k\sqrt{x^2+y^2}$ funksiya bilan aniqlanar ekan, bu yerda $k-$ proporsionallik koffisiyenti. Figuraning m massasini (8.46) formulaga ko‘ra hisoblaymiz:



8.39-rasm

$$m=\iint_D \mu(x,y)dxdy=k\iint_D \sqrt{x^2+y^2}dxdy.$$

Integrallash sohasining shakli va integral ostidagi funksiyaning ko‘rinishidan kelib chiqib $x=ar\cos\varphi$, $y=ar\sin\varphi$ deb qutb koordinatalar sistemasiga o‘tamiz. Aylana tenglamasining ko‘rinishini o‘zgartiramiz:

$$x^2-2\cdot\frac{a}{2}\cdot x+\frac{a^2}{4}+y^2=\frac{a^2}{4},$$

yoki

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4},$$

qutb koordinatalariga o‘tsak

$$\left(ar\cos\varphi-\frac{a}{2}\right)^2+(ar\sin\varphi)^2=\frac{a^2}{4}.$$

So‘ngi tenglamani ixchamlasak

$$r=\cos\varphi \tag{8.47}$$

tenglamani hosil qilamiz. D soha yarim aylana bilan chegaralanganligi uchun φ qutb burchagi $[0,\pi/2]$ kesmada o‘zgaradi va o‘zgarmas $\varphi\in[0,\pi/2]$ burchakda (8.47) tenglikka ko‘ra r qutb radiusi $[0,\cos\varphi]$ kesmada o‘zgaradi. Shuning uchun figura massasi uchun

$$m=k\iint_D \sqrt{x^2+y^2}dxdy=k\int_0^{\pi/2}d\varphi\int_0^{\cos\varphi}ara^2rdr=$$

$$=ka^3\int_0^{\pi/2}d\varphi\int_0^{\cos\varphi}r^2dr=ka^3\int_0^{\pi/2}\left(\frac{1}{3}r^3\Big|_0^{\cos\varphi}\right)d\varphi=\frac{ka^3}{3}\int_0^{\pi/2}\cos^3\varphi\,d\varphi=$$

$$=\frac{ka^3}{3}\int_0^{\pi/2}(1-\sin^2\varphi)d\sin\varphi=\frac{ka^3}{3}\left(\sin\varphi-\frac{1}{3}\sin^3\varphi\right)\Big|_0^{\pi/2}=\frac{2}{9}ka^3$$

tenglikni hosil qilamiz.◄

Yassi figuraning koordinata o‘qlariga nisbatan statik va inersiya momentlari. Massasi m bo‘lgan material nuqtaning Ox o‘qqa nisbatan S_x statik momenti deb my ko‘paytmaga aytiladi, bu yerda y −material nuqtaning ordinatasi, ya’ni

$$S_x=my. \tag{8.48}$$

$y$ordinataning qiymati musbat bo‘lishi ham, manfiy bo‘lishi ham mumkin.

Bu nuqtaning Ox o‘qqa nisbatan I_x inersiya momenti deb my^2 ko‘paytmaga aytiladi:

$$I_x=my^2 \tag{8.49}$$

Massalari m_1, m_2,…,m_k bo‘lgan M_1, M_2,…,M_k nuqtalar sistemasining Ox o‘qqa nisbatan S_x statik va I_x inersiya momentlari deb, bu nuqtalarning mos ravishda statik va inersiya momentlari yig‘indisiga aytiladi:

$$S_x=\sum_{i=1}^{k}m_iy_i,\quad I_x=\sum_{i=1}^{k}m_iy_i^2 \tag{8.50}$$

Oxy tekislikda sirt zichligi uzluksiz $\mu(x,y)$ funksiya bilan aniqlangan yassi D figuraning Ox o‘qqa nisbatan statik va inersiya momentlarini hisoblaymiz.

D sohani n ta D_1, D_2,…,D_n qismlarga ajratib, har bir D_i qismda ixtiyoriy $P_i(x_i,y_i)$ nuqtani tanlaymiz va i −qismning massasi taqriban $\mu(x_i,y_i)\Delta S_i$ ga teng va bu massa $P_i(x_i,y_i)$ nuqtada jamlangan deb hisoblaymiz, bu yerda ΔS_i qismiy D_i sohaning yuzi. U holda D yassi figuraning Ox o‘qqa nisbatan S_x statik va I_x inersiya momentlari taqriban

$$S_x\approx\sum_{i=1}^{k}y_i\mu(x_i,y_i)\Delta S_i,\qquad I_x\approx\sum_{i=1}^{k}y_i^2\mu(x_i,y_i)\Delta S_i$$

yig‘indilarga teng bo‘ladi. Bu taqribiy tengliklarda D_i qismiy sohalar diametrlaring d eng kattasini nolga intiltirib va bu yig‘indilar mos ravishda $y\mu(x,y)$ va $y^2\mu(x,y)$ funksiyalarning integral yig‘indilari ekanligini inobatga olib

$$S_x=\iint_D y\mu(x,y)dxdy \tag{8.51}$$

$$I_x=\iint_D y^2\mu(x,y)dxdy \tag{8.52}$$

formulalarni hosil qilamiz.

Xuddi shu singari sirt zichligi $\mu(x,y)$ funksiya bilan aniqlanadigan yassi figuraning Oyo‘qqa nisbatan statik va inersiya momentlari mos ravishda

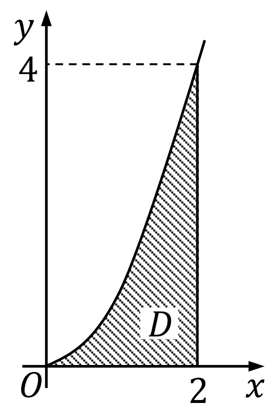
$$S_y = \iint_D x\mu(x,y)dxdy \tag{8.53}$$

$$I_y = \iint_D x^2\mu(x,y)dxdy \tag{8.54}$$

formulalar bilan aniqlanadi.

8.13-Misol. Yuqoridan $y = x^2$ parabola bilan, quyidan Ox o‘q bilan va yon tomondan $x = 2$ to‘g‘ri chiziq bilan chegaralangan yassi figuraning koordinata o‘qlariga nisbatan statik va inersiya momentlarini toping (8.40-rasm). Sirt zichligi har bir nuqtada nuqta koordinatalarining ko‘paytmasiga proporsional.

►Shartga ko‘ra $\mu(x,y) = k \cdot xy$, bu yerda $k -$ proporsionallik koffisiyenti. Koordinata o‘qlariga nisbatan statik va inersiya momentlarini (8.51)-(8.54) formulalar bo‘yicha topamiz:



8.40-rasm

$$S_x = \iint_D y\mu(x,y)dxdy = \iint_D y \cdot k \cdot xydxdy =$$

$$= k\iint_D xy^2dxdy = k\int_0^2 dx\int_0^{x^2} xy^2dy =$$

$$= k\int_0^2 \left(\frac{xy^3}{3}\bigg|_0^{x^2}\right)dx = k\int_0^2 \frac{x^7}{3}dx = k\frac{x^8}{24}\bigg|_0^2 = k\frac{32}{3},$$

$$S_y = \iint_D x\mu(x,y)dxdy = \iint_D x \cdot k \cdot xydxdy =$$

$$= k\iint_D x^2ydxdy = k\int_0^2 dx\int_0^{x^2} x^2ydy = k\int_0^2 \left(\frac{x^2y^2}{2}\bigg|_0^{x^2}\right)dx =$$

$$= k\int_0^2 \frac{x^6}{2}dx = k\frac{x^7}{14}\bigg|_0^2 = k\frac{64}{7},$$

$$I_x = \iint_D y^2\mu(x,y)dxdy = \iint_D y^2k \cdot xydxdy = k\iint_D xy^3dxdy =$$

$$= k\int_0^2 dx\int_0^{x^2} xy^3dy = k\int_0^2 \left(\frac{xy^4}{4}\bigg|_0^{x^2}\right)dx = k\int_0^2 \frac{x^9}{4}dx = k\frac{x^{10}}{40}\bigg|_0^2 = k\frac{128}{5},$$

$$I_y = \iint_D x^2\mu(x,y)dxdy = \iint_D x^2k \cdot xydxdy = k\iint_D x^3ydxdy =$$

$$= k\int_0^2 dx\int_0^{x^2} x^3y dy \ = k\int_0^2 \left(\frac{x^3y^2}{2}\bigg|_0^{x^2}\right)dx \ = k\int_0^2 \frac{x^7}{2}dx \ = k\frac{x^8}{16}\bigg|_0^2 = 16k. \quad ◀$$

Yassi figura og‘irlik markazining koordinatalari. Agar yassi figuraning koordinata o‘qlariga nisbatan S_x va S_y statik momentlari va m massasi ma’lum bo‘lsa, bu figura *og‘irlik markazining* koordinatalari

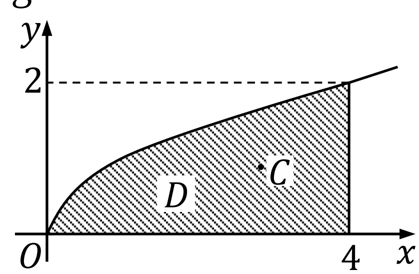
$$x_c = \frac{S_y}{m}, \quad y_c = \frac{S_x}{m} \qquad (8.55)$$

formulalarga ko‘ra topiladi. Bu formulalarga (8.46), (8.51) va (8.53) formulalarni qo‘llab

$$x_c = \frac{\iint\limits_D x\mu(x,y)dxdy}{\iint\limits_D \mu(x,y)dxdy}, \qquad y_c = \frac{\iint\limits_D y\mu(x,y)dxdy}{\iint\limits_D \mu(x,y)dxdy} \qquad (8.56)$$

o‘g‘irlik markazining koordinatalarini topish formulasini hosil qildik.

8.14-Misol. Yuqoridan $y = \sqrt{x}$ egri chiziq bilan, quyidan Oxkoordinata o‘qi bilan va yon tomondan $x = 4$ to‘g‘ri chiziq bilan chegaralangan D yassi figuraning og‘irlik markazi koordinatalarini toping. Sirt zichligi har bir nuqtada nuqta koordinatalarining yig‘indisiga proporsional (8.41-rasm).



8.41-rasm

► Shartga ko‘ra $\mu(x,y) = k \cdot (x+y)$, bu yerda $k -$ proporsionallik koffisiyenti. D yassi figuraning massasini va koordinata o‘qlariga nisbatan statik momentlarini (8.46),(8.51),(8.53) formulalar bo‘yicha topamiz:

$$m = \iint\limits_D \mu(x,y)dxdy = \iint\limits_D k\cdot(x+y)dxdy = k\iint\limits_D (x+y)dxdy =$$

$$= k\int_0^4 dx\int_0^{\sqrt{x}} (x+y)dy \ = k\int_0^4 \left[\left(xy + \frac{1}{2}y^2\right)\bigg|_0^{\sqrt{x}}\right]dx \ = k\int_0^4 \left(x^{3/2} + \frac{1}{2}x\right)dx \ =$$

$$= k\left(\frac{2}{5}x^{5/2} + \frac{1}{4}x^2\right)\bigg|_0^4 = k\left(\frac{2}{5}\cdot 32 + 4\right) = \frac{84}{5}k,$$

$$S_x = \iint\limits_D y\mu(x,y)dxdy = \iint\limits_D y\cdot k\cdot(x+y)dxdy = k\iint\limits_D (xy + y^2)dxdy =$$

$$= k\int_0^4 dx\int_0^{\sqrt{x}} (xy + y^2)dy = k\int_0^4 \left[\left(\frac{xy^2}{2} + \frac{1}{3}y^3\right)\Big|_0^{\sqrt{x}}\right]dx =$$

$$= k\int_0^4 \left(\frac{x^2}{2} + \frac{1}{3}y^{3/2}\right)dx \;= k\left(\frac{x^3}{6} + \frac{2}{15}x^{5/2}\right)\Big|_0^4 = k\left(\frac{64}{6} + \frac{2}{15}\cdot 32\right) = \frac{224}{15}k,$$

$$S_y = \iint_D x\mu(x,y)dxdy = \iint_D x\cdot k\cdot(x+y)dxdy = k\iint_D (x^2 + xy)dxdy =$$

$$= k\int_0^4 dx\int_0^{\sqrt{x}} (x^2 + xy)dy = k\int_0^4 \left[\left(x^2y + \frac{1}{2}xy^2\right)\Big|_0^{\sqrt{x}}\right]dx =$$

$$= k\int_0^4 \left(x^{5/2} + \frac{1}{2}x^2\right)dx \;= k\left(\frac{2}{7}x^{7/2} + \frac{1}{6}x^3\right)\Big|_0^4 = k\left(\frac{2}{7}\cdot 128 + \frac{64}{6}\right) = \frac{992}{21}k.$$

(8.55) formulalarga ko‘ra D yassi figura $C(x_c, y_c)$ og‘irlik markazi koordinatalarini topamiz:

$$x_c = \frac{S_y}{m} = \frac{992k}{21} : \frac{84k}{5} = \frac{1240}{441} \approx 2{,}81,$$

$$y_c = \frac{S_x}{m} = \frac{224k}{15} : \frac{84k}{5} = \frac{56}{63} \approx 0{,}89.$$

Shunday qilib qaralayotgan D figuraning og‘irlik markazi $C\left(\frac{1240}{441}, \frac{56}{63}\right)$ nuqtada bo‘lar ekan.◄

8.7. Uch o‘lchovli integral

Ikki o‘lchovli integralni umumlashtirish orqali hosil qilinadigan uch o‘zgaruvchili funksiyaning *uch o‘lchovli integralini* qarab chiqamiz. Bunday integrallar ikki o‘lchovli integrallar singari turli geometrik, fizik va texnik masalalrni yechishda tadbiq qilinadi. Ikki va uch o‘lchovli integrallar orasida to‘liq o‘xshashlik mavjudligi tufayli biz faqat tasdiqlarni bayon qilish bilan chegaralanamiz. Ularning isbotini ikki o‘lchovli integrallardagi isbotlarni uch o‘zgaruvchili funksiyalarga osongina moslashtirish orqali hosil qilish mumkin.

Jismning massasini hisoblash haqida masala. Fazoviy Ω sohani massa bilam to‘ldirilgan jism berilgan bo‘lsin. Agar har bir $P \in \Omega$ nuqtada

$$\mu = \mu(P) = \mu(x,y,z),\;\; P(x,y,z) \in \Omega$$

massa taqsimlanishining zichligi berilgan bo‘lsa, bu jismning m massasini topish talab qilingan bo‘lsin.

Ω sohani qandaydir sirtlar bilan hajmlari ΔV_1, ΔV_2,…, ΔV_n bo‘lgan n ta $\Delta\Omega_1$, $\Delta\Omega_2$, …, $\Delta\Omega_n$ qismlarga ajratamiz. Har bir $\Delta\Omega_i$ sohada ixtiyoriy ravishda bittadan $P_i(x_i, y_i, z_i)$ nuqtani tanlaymiz.$\Delta\Omega_i$sohada zichlik taqriban o‘zgarmas va u $\mu(P_i)$ ga teng deb faraz qilamiz. U holda bu bo‘lakning massasini

$$\Delta m_i \approx \mu(P_i)\Delta V_i$$

taqribiy tenglik bilan aniqlash mumkin, butun jismning massasi esa taqriban

$$m \approx \sum_{i=1}^{k} \mu(P_i)\Delta V_i \tag{8.57}$$

yig‘indiga teng bo‘ladi.

$\Delta\Omega_i$ bo‘laklarning o‘lchamlari qanchalik kichik bo‘lsa, (8.57) taqribiy tenglik shunchalik aniqroq bo‘ladi. Qaralayotgan jismning massasi sifatida (8.57) tenglik o‘ng tomonidagi yig‘indining $\Delta\Omega_i$ $(i = 1,2, \dots, n)$ bo‘laklar diametrlarining eng kattasi d nolga intilgandagi limitini olish mumkin:

$$m = \lim_{d\to 0} \sum_{k=1}^{n} \mu(P_i)\Delta V_i. \tag{8.58}$$

Uch o‘lchovli integral. Fazoda S sirt bilan chegaralangan Ω soha berilgan va bu soha va uning S chegarasida uzluksiz $f(x, y, z)$ funksiya aniqlangan bo‘lsin.

Ω sohani qandaydir sirtlar bilan hajmlari ΔV_1, ΔV_2,…, ΔV_n bo‘lgan n ta $\Delta\Omega_1$, $\Delta\Omega_2$, …, $\Delta\Omega_n$ qismlarga ajratamiz. Bu bo‘laklar diametrlarini d_i orqali va ularning eng kattasini d orqali belgilaymiz. Har bir $\Delta\Omega_i$ sohada ixtiyoriy ravishda bittadan $P_i(x_i, y_i, z_i)$ nuqtani tanlaymiz va$f(x, y, z)$ funksiyaning bu nuqtalardagi $f(x_i, y_i, z_i)$ qiymatlarini hisoblab:

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i \tag{8.59}$$

yig‘indini tuzamiz. Bu yig‘indi $f(x, y, z)$ funksiyaning Ω soha bo‘yicha *integral yig‘indisi* deb ataladi.

8.3-Ta’rif. Agar (8.59) integral yig‘indi $d = 0$ nuqtada chekli I limitga ega bo‘lib, u Ω sohani $\Delta\Omega_1$, $\Delta\Omega_2$, …, $\Delta\Omega_n$ qismlarga bo‘lish usuliga va ulardagi P_i nuqtalarning tanlanishiga bog‘liq bo‘lmasa, bu limitni biz $f(x, y, z)$ funksiyaning Ω soha bo‘yicha olingan uch o‘lchovli integrali deb ataymiz va uni

$$I = \iiint_{\Omega} f(x, y, z)dV = \iiint_{\Omega} f(x, y, z)dxdydz$$

orqali belgilaymiz.

Shunday qilib, uch o‘lchovli integral

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \lim_{d\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i \qquad (8.60)$$

tenglik bilan aniqlanar ekan. Bunday holda $f(x,y,z)$ funksiya Ω sohada *integrallanuvchi* va Ω −*integrallash sohasi* deb ataladi.

(8.58) va (8.60) tengliklarni taqqoslab, nuqtalaridagi zichligi $\mu(x,y,z)$ funksiya bilan aniqlanuvchi Ω jismning m massasi uchun

$$m = \iiint\limits_{\Omega} f(x,y,z)dxdydz \qquad (8.61)$$

formulani hosil qilamiz.

Har qanday $f(x,y,z)$ funksiya uchun ham uch o‘lchovli integral mavjudmi degan savolga quyidagi teorema javob beradi.

8.4-Teorema. Yopiq chegaralanagan Ω sohada uzluksiz $f(x,y,z)$ funksiya, bu sohada integrallanuvchi hamdir.

Biz quyida integrallash sohasida faqat uzluksiz bo‘lgan funksiyalarnigina qaraymiz.

Uch o‘lchovli integralning xossalari. Uch o‘lchovli integralning xossalri ikki o‘lchovli integralning xossalariga o‘xshashligi tufayli, biz ularni bayon qilish bilangina chegaralanamiz.

1) Agar Ω soha V hajmga ega bo‘lsa, u holda

$$\iiint\limits_{\Omega} dxdydz = V. \qquad (8.62)$$

2) Agar $f(x,y,z)$ va $g(x,y,z)$ funksiyalar Ω sohada integrallanuvchi bo‘lsa, u holda ularning ixtiyoriy α va β o‘zgarmaslar bilan chiziqli $\alpha f(x,y,z) + \beta g(x,y,z)$ kombinatsiyasi ham Ω sohada integrallanuvchi bo‘ladi va

$$\iiint\limits_{\Omega} (\alpha f(x,y,z) + \beta g(x,y,z))dxdydz =$$
$$= \alpha\iiint\limits_{\Omega} f(x,y,z)dxdydz + \beta\iiint\limits_{\Omega} g(x,y,z)dxdydz \qquad (8.63)$$

tenglik o‘rinli bo‘ladi. Bu xossa uch o‘lchovli integralninmg *chiziqlilik xossasi* deb ataladi.

3) Agar Ω soha $\Omega_1 \cup \Omega_2 = \Omega$ bo‘ladigan qilib faqat bo‘linish sirtlaridagina kesishadigan ikkita Ω_1 va Ω_2 sohalarga ajratilsa,

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \iiint\limits_{\Omega_1} f(x,y,z)dxdydz +$$
$$+\iiint\limits_{\Omega_2} f(x,y,z)dxdydz \qquad (8.64)$$

tenglik o‘rinli bo‘ladi.

4) Agar $f(x,y,z)$ funksiya Ω sohada nomanfiy va integrallanuvchi bo‘lsa,

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz \geq 0 \qquad (8.65)$$

bo‘ladi.

5) Agar $f(x,y,z)$va$g(x,y,z)$ funksiyalar Ω sohada integrallanuvchi va barcha $(x,y,z) \in \Omega$ nuqtalarda $f(x,y,z) \geq g(x,y,z)$ bo‘lsa,

$$\iiint\limits_{D} f(x,y,z)dxdydz \geq \iiint\limits_{D} g(x,y,z)dxdydz \qquad (8.66)$$

bo‘ladi.

6) Agar $f(x,y,z)$ funksiya Ω sohada integrallanuvchi bo‘lsa, $|f(x,y)|$ funksiya ham Ω sohada integrallanuvchi bo‘ladi va

$$\left|\iint\limits_{\Omega} f(x,y,z)dxdydz\right| \geq \iint\limits_{\Omega} |f(x,y,z)|dxdydz \qquad (8.67)$$

tengsizlik o‘rinli bo‘ladi.

7) Agar hajmi V ga teng bo‘lganΩ sohada $f(x,y,z)$ funksiya uzluksiz va uning bu sohadagi eng katta va eng kichik qiymatlari mos ravishda M va m bo‘lsa,

$$mV \leq \iiint\limits_{\Omega} f(x,y,z)dxdydz \leq MV \qquad (8.68)$$

tengsizlilar o‘rinli bo‘ladi.

8) (O‘rta qiymat haqidagi teorema) Agar hajmi V ga teng bo‘lgan Ω sohada $f(x,y,z)$ funksiya uzluksiz bo‘lsa, bu sohada

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = f(x_0,y_0,z_0)\cdot V$$

tenglikni qanoatlantiruvchi $P_0(x_0,y_0,z_0)$ nuqta mavjud.

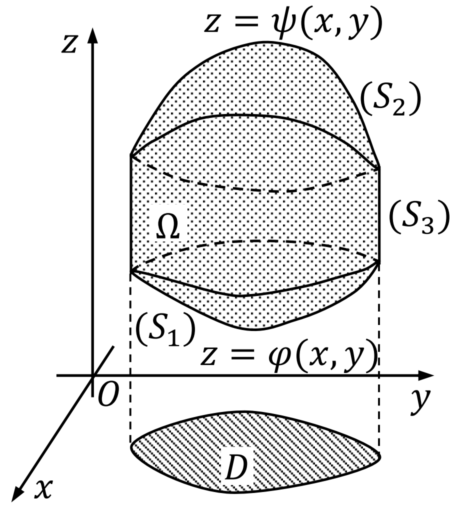
$$f(x_0,y_0,z_0) = \frac{1}{V}\iint\limits_{\Omega} f(x,y,z)dxdydz \qquad (8.69)$$

miqdor $f(x,y,z)$ funksiyaning Ω sohadagi *o‘rta qiymati* deb ataladi.

Uch o‘lchovli integralni hisoblash. Ikki o‘lchovli integrallardagi singari uch o‘lchovli integralni hisoblash takroriy integralni hisoblashga keltiriladi.

Fazodagi Ω soha quyidan $S_1: z = \varphi(x,y)$ sirt bilan, yuqoridan $S_2: z = \psi(x,y)$ sirt bilan, Oz o‘qqa parallel yon silindrik S_3 sirt bilan chegaralangan bo‘lsin, bu yerda $(x,y) \in D$ nuqtalarda $\varphi(x,y) \leq \psi(x,y)$ tengsizlik o‘rinli va $D -$ soha berilgan fazoviy Ω sohaning Oxy tekislikdagi proyeksiyasi bo‘lsin (8.42-rasm). Agar Ω sohaning ixtiyoriy ichki nuqtasidan Oz o‘qqa parallel bo‘lib o‘tuvchi to‘g‘ri chiziq, bu sohaning chegarasini faqat ikki nuqtada kesib o‘tsa, bunday soha z *−silindrik soha* yoki Oz *o‘qi yo‘nalishi bo‘yicha to‘g‘ri soha* deb ataladi.

Ikki o‘lchovli integrallar uchun keltirilgan 8.3-Teoremaga o‘xshash quyidagi teorema o‘rinli.



8.42-rasm

8.4-Teorema. Agar $f(x,y,z)$ funksiyaning $\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, \varphi(x,y) \leq z \leq$ $\leq \psi(x,y)$ soha bo‘yicha uch o‘lchovli integrali mavjud va har bir $(x,y) \in D$ nuqta uchun

$$I = \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz$$

aniq integral mavjud bo‘lsa, u holda

$$\iint_D dxdy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \tag{8.70}$$

takroriy integral ham mavjud va

$$\iiint_\Omega f(x,y,z)dxdydz = \iint_D dxdy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \tag{8.71}$$

tenglik o‘rinli bo‘ladi.

Mulohaza. Agar 8.4-Teoremadagi I integral uchun 8.3-Teoremaning shartlari o‘rinli bo‘lsa, uni dastlab y bo‘yicha, so‘ngra x bo‘yicha olinadigan takroriy integral ko‘rinishida tasvirlash mumkin bo‘ladi. Bu holda (8.71) tenglik

$$\iiint_\Omega f(x,y,z)dxdydz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \tag{8.72}$$

ko‘rinishni oladi.

Agar Ω fazoviy soha Oy o‘qi yoki Ox o‘qi yo‘nalishi bo‘yicha to‘g‘ri soha bo‘lsa, (8.71) formulada x, y, z o‘zgaruvchilarning o‘rni almashtirilib, uch o‘lchovli integralni boshqa tartibdagi takroriy integralga keltirish mumkin.

Mulohaza. Agar Ω fazoviy soha hech bir koordinata o‘qi yo‘nalishi bo‘yicha to‘g‘ri bo‘lmasa, bu sohani har biri hech bo‘lmasa bitta o‘q yo‘nalishi bo‘yicha to‘g‘ri bo‘ladigan qismlarga ajratish kerak. So‘ngra har bir bo‘lakka 8.4-Teoremani qo‘llab integral hisoblanadi va uch o‘lchovli integrallarning 3-xossasi qo‘llanilsa, hisoblangan integrallarning yig‘indisi berilgan uch o‘lchovli integralning qiymatini beradi.

8.15-Misol. $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x \in [a,b], y \in [c,d], z \in [0,h]\}$ soha bo‘yicha integrallash chegaralarini qo‘yamiz.

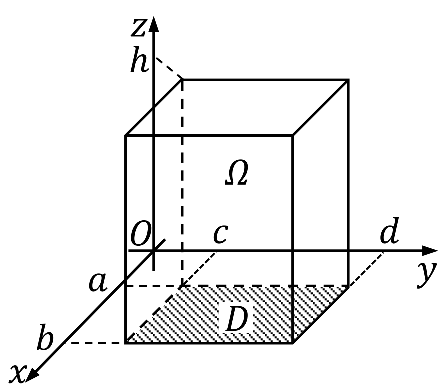
►Mazkur holda $\Omega -$ qirralari koordinata o‘qlariga parallel bo‘lgan to‘g‘ri burchakli parallelepipeddan iborat (8.43-rasm). Bu parallelepiped barcha koordinata o‘qlari yo‘nalishi bo‘yicha to‘g‘ri yopiq soha. Vertikal to‘g‘ri chiziqlar uning quyi va yuqori yoqlarini $z_1 = 0$ va $z_2 = h$ qiymatlarda kesib o‘tadi. Shuning uchun (8.71) formulaga ko‘ra

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \iint\limits_{D} dxdy \int\limits_{0}^{h} f(x,y,z)dz$$

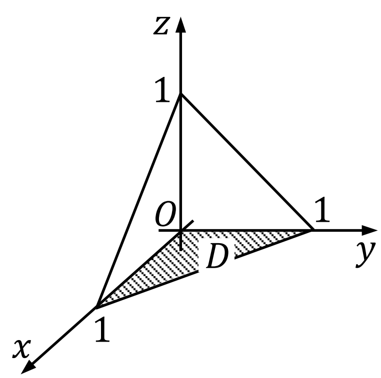
tenglik o‘rinli bo‘ladi, bu yerda $D = \{(x,y) \in \boldsymbol{R}^2 : x \in [a,b], y \in [c,d]\}$ integrallash sohasi Oxy tekislikdagi to‘g‘ri to‘rtburchak. 8.2-Teoremani qo‘llab

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \int\limits_{a}^{b} dx \int\limits_{c}^{d} dy \int\limits_{0}^{h} f(x,y,z)dz$$

tenglikni hosil qilamiz.◄



8.43-rasm



8.44-rasm

8.16-Misol. $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$ tetraedr bo‘yicha integrallash chegaralarini qo‘yamiz (8.44-rasm).

► Ω integrallash sohasi barcha koordinata o‘qlari yo‘nalishi bo‘yicha to‘g‘ri soha. Oxy tekislikning(x,y) nuqtasidan o‘tuvchi to‘g‘i chiziq Ω yopiq sohani quyidan $(x,y,0)$ nuqtada va yuqorida $x + y + z = 1$ tekislikning $(x,y,1-x-y)$ nuqtasida kesib o‘tadi. Ω sohaning Oxy tekislikka proyeksiyasi

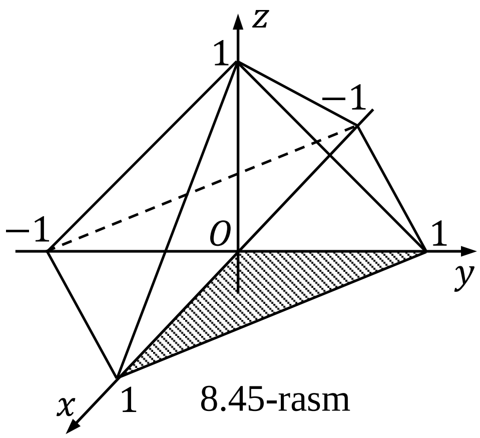
$$D = \{(x,y) \in \mathbb{R}^2 : x \in [0,1], x + y \leq 1\}$$

uchburchak bo‘ladi. Bu D sohada Oy o‘qqa parallel to‘g‘ri chiziqlar uning chegarasini $y = 0$ va $y = 1 - x$ to‘g‘ri chiziqlarda kesib o‘tadi. Shuning uchun (8.72) formulani va 8.3-Teoremani qo‘llasak, uch o‘lchovli integralni

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} f(x,y,z)dz$$
koʻrinishda tasvirlash mumkin.◄

8.17-Misol. Uchlari Ox va Oy oʻqlarning ham manfiy ham musbat tomonlarida, Oz oʻqda esa faqat musbat tomonida koordinata boshidan bir birlik uzoqlikda yotuvchi nuqtalarda boʻlgan Ω −piramidaning hajmini toping (8.45-rasm).

►Mazkur piramidaning Oxz va Oyz tekisliklarga nisbatan simmetrikligi tufayli, pirmidaning birinchi oktantdagi qismining hajmini hisoblaymiz va natijani toʻrtga koʻpaytirib qidirilayotgan hajmni topamiz. Birinchi oktantdagi qism uchun uch oʻlchovli integral 8.16-Misolda qaraldi. Shunday qilib



8.45-rasm

$$V = \iiint\limits_{\Omega} dxdydz = 4\int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} dz = 4\int\limits_0^1 dx \int\limits_0^{1-x} \left[z|_0^{1-x-y}\right]dy =$$
$$= 4\int\limits_0^1 dx \int\limits_0^{1-x} (1-x-y)dy = 4\int\limits_0^1 \left((1-x)y - \frac{1}{2}y^2\right)\Bigg|_0^{1-x} dx =$$
$$= 4\int\limits_0^1 \frac{1}{2}(1-x)^2dx = -\frac{2}{3}(1-x)^3\Big|_0^1 = \frac{2}{3}. \quad ◄$$

8.8. Uch oʻlchovli integralda oʻzgaruvchilarni almashtirish

Fazoda egri chiziqli koordinatalar. $Oxyz$ fazoning Ω sohasi $Ouvw$ sohaning G sohasiga

$$\begin{cases} u = p(x,y,z), \\ v = q(x,y,z), \\ w = r(x,y,z) \end{cases} \tag{8.73}$$

formulalar yordamida akslantirilsin. Bunda Ω sohaning $\partial\Omega$ chegara sirti G sohaning ∂G chegara sirtiga akslantiriladi.

(8.73) akslantirish o‘zaro bir qiymatli bo‘lganligi tufayli ixtiyoriy $(u, v, w) \in G$ nuqta uchun

$$\begin{cases} x = g(u, v, w), \\ y = h(u, v, w), \\ z = l(u, v, w) \end{cases} \tag{8.74}$$

tengliklarni qanoatlantiruvchi g, h, l funksiyalarni topish mumkin.

Demak, (u, v, w) sonlar uchligining berilishi (x, y, z) sonlar uchligining berilishi bilan teng kuchli ekan, shu sababli u, v va w qiymatlarni (x, y, z) nuqtaning fazodagi egri chiziqli koordinatalari deb ataladi.

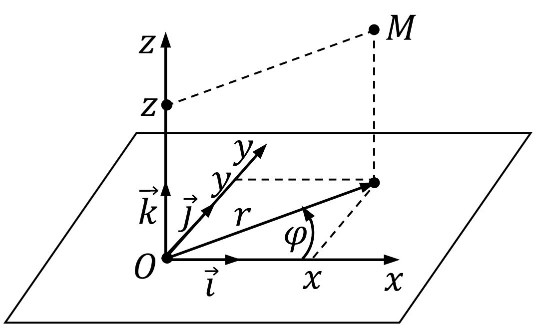
Egri chiziqli koordinatalardan bittasi o‘zgarmas bo‘ladigan nuqtalarning geometrik o‘rniga bu koordinataning koordinata sirti deb ataladi. u, v va w koordinatalarning har biri uchun koordinata sirtlarining butun bir oilalari mavjud bo‘lishi ravshan. Bunda Ω sohaning har bir nuqtasi orqali har bir oilaning bitta koordinata sirti o‘tadi.

Fazodagi egri chiziqli koordinatalarga misol sifatida silindrik koordinatalarni ko‘rsatish mumkin. Ular *qutb radiyus* $(r \geq 0)$, *qutb burchak* $(-\pi < \varphi \leq \pi)$ va *applikata* $(-\infty < z < +\infty)$ deb nomlanadi va fazodagi M nuqtaning holatini to‘liq tavsiflaydi (8.46-rasm).

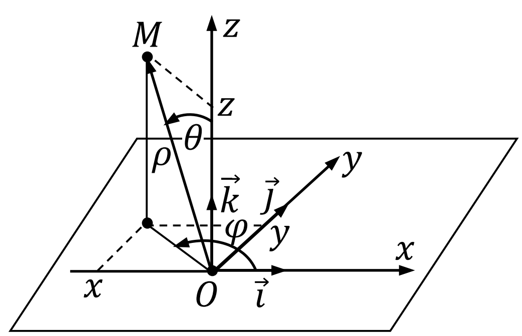
(8.74) formulalar silindrik koordinatalar uchun

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi\,, \\ z = z \end{cases} \tag{8.75}$$

ko‘rinishda bo‘ladi.



8.46-rasm



8.47-rasm

Silindrik koordinatalar uchun koordinata sirtlari:

i) r qutb radiusi uchun applikata o‘qiga parallel bo‘lgan to‘g‘ri doiraviy silindrlardan iborat bo‘ladi;

ii) φ qutb burchagi uchun applikata o‘qi bilan chegaralangan yarim tekisliklardan iborat bo‘ladi;

iii) z applikata uchun applikata o‘qiga perpendikulyar bo‘lgan tekisliklardan iborat bo‘ladi.

Fazodagi egri chiziqli koordinatalarga yana bir misol sifatida sferik koordinatar deb atalauvchi ρ, θ, φ uchlikni ko‘rsatish mumkin, bunda ular *radius* $(\rho \geq 0)$, *kenglik*$(0 \leq \theta \leq \pi)$ va *uzoqlik*$(-\pi < \varphi \leq \pi)$ deb nomlanadi (8.47-rasm).

(8.74) formulalar sferik koordinatalar uchun

$$\begin{cases} x = \rho \sin\theta \cos\varphi, \\ y = \rho \sin\theta \sin\varphi, \\ z = \rho \cos\theta. \end{cases} \tag{8.76}$$

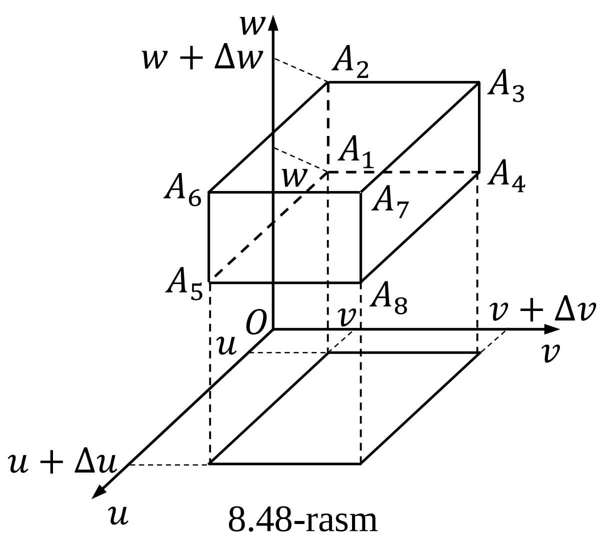
ko‘rinishda bo‘ladi.

Sferik koordinatalar uchun koordinata sirtlari:

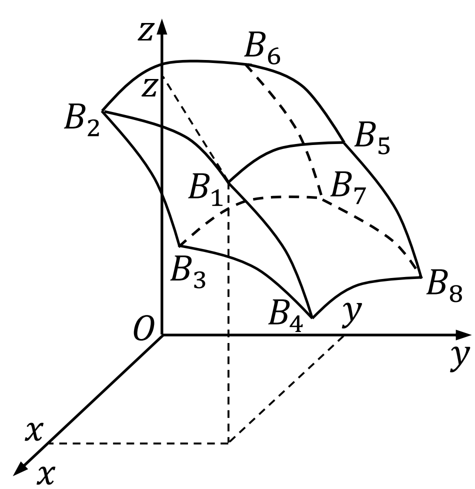
i) ρ radius uchun markazi koordinatalar boshida bo‘lgan sferalardan iborat bo‘ladi;

ii) θ kenglik uchun uchi koordinatalar boshida va o‘qi applikatalar o‘qi bilan ustma-ust tushadigan konuslardan iborat bo‘ladi;

iii) φ uzoqlik uchun applikatalar o‘qi bilan chegaralangan yarim tekisliklardan iborat bo‘ladi.



8.48-rasm



8.49-rasm

Egri chiziqli koordinatalarda hajm elementi. G sohada Ouv, Ouw, Ovw koordinata tekisliklariga parallel bo‘lgan uch juft tekisliklardan hosil qilingan va qirralari Δu, Δv, Δw bo‘lgan elementar to‘g‘ri to‘rt burchakli parallelepipedni olamiz (8.48-rasm). Bu parallelepipedning hajmi $\Delta\omega = \Delta u \cdot \Delta v \cdot \Delta w$ tenglik bilan hisoblanadi.

(8.74) akslantirishda bu parallelepiped umumiy holda egri chiziqli olti yoqqa o‘tadi (8.49-rasm). Bunda parallelepipedning $u = \text{const}$ tekislikda yotuvchi $A_1A_2A_3A_4$ yoqiga Ω sohadagi $B_1B_2B_3B_4$ sirt mos keladi, $u + \Delta u -$tekislikda yotuvchi $A_5A_6A_7A_8$ to‘g‘ri to‘rt burchakka $B_5B_6B_7B_8$ sirt mos keladi va hokazo. Gsohadagi parallelepipedning o‘lchovlari yetarlicha kichik bo‘lganda unga mos keluvchi oltiyoqni ham taqriban $\overrightarrow{B_1B_5}$, $\overrightarrow{B_1B_2}$ va $\overrightarrow{B_1B_4}$ vektorlardan yasalgan parallelepiped deb qarash mumkin. Endi bu vektorlarni (8.74) akslantirishlar orqali ifodalaymiz.

$\overrightarrow{B_1B_5}$ vektor Δu orttirmaga mos keladi va shu sababli uning koordinatalari uchun

$$\overrightarrow{B_1B_5} \approx \left\{\frac{\partial g}{\partial u}\Delta u, \frac{\partial h}{\partial u}\Delta u, \frac{\partial l}{\partial u}\Delta u\right\}$$

taqribiy tenglikni hosil qilamiz. Xuddi shu singari $\overrightarrow{B_1B_2}$ va $\overrightarrow{B_1B_4}$ vektorlar uchun

$$\overrightarrow{B_1B_2} \approx \left\{\frac{\partial g}{\partial v}\Delta v, \frac{\partial h}{\partial v}\Delta v, \frac{\partial l}{\partial v}\Delta v\right\},$$

$$\overrightarrow{B_1B_4} \approx \left\{\frac{\partial g}{\partial w}\Delta w, \frac{\partial h}{\partial w}\Delta w, \frac{\partial l}{\partial w}\Delta w\right\}$$

taqribiy tengliklarni yozish mumkin. Ma’lumki $\overrightarrow{B_1B_5}$, $\overrightarrow{B_1B_2}$ va $\overrightarrow{B_1B_4}$ vektorlardan tuzilgan parallelepipedning ΔV hajmi bu vektorlar aralash ko‘paytmasining absolyut qiymatiga teng:

$$\Delta V = \left|\overrightarrow{B_1B_5}\overrightarrow{B_1B_2}\overrightarrow{B_1B_4}\right| \approx \begin{vmatrix} \frac{\partial g}{\partial u}\Delta u & \frac{\partial h}{\partial u}\Delta u & \frac{\partial l}{\partial u}\Delta u \\ \frac{\partial g}{\partial v}\Delta v & \frac{\partial h}{\partial v}\Delta v & \frac{\partial l}{\partial v}\Delta v \\ \frac{\partial g}{\partial w}\Delta w & \frac{\partial h}{\partial w}\Delta w & \frac{\partial l}{\partial w}\Delta w \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} & \frac{\partial l}{\partial u} \\ \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} & \frac{\partial l}{\partial v} \\ \frac{\partial g}{\partial w} & \frac{\partial h}{\partial w} & \frac{\partial l}{\partial w} \end{vmatrix} \Delta u \Delta v \Delta w.$$

So‘ngi tenglikdagi

$$J(u,v,w)=\begin{vmatrix}\dfrac{\partial g}{\partial u} & \dfrac{\partial h}{\partial u} & \dfrac{\partial l}{\partial u}\\ \dfrac{\partial g}{\partial v} & \dfrac{\partial h}{\partial v} & \dfrac{\partial l}{\partial v}\\ \dfrac{\partial g}{\partial w} & \dfrac{\partial h}{\partial w} & \dfrac{\partial l}{\partial w}\end{vmatrix}$$

determinant $g(u,v,w)$, $h(u,v,w)$ va $l(u,v,w)$ funksiyalarning yakobiyani deb ataladi. Shunday qilib yuqorida aytib o‘tilgan og‘ma parallelepiped hajmi

$$\Delta V=|J(u,v,w)|\Delta u\Delta v\Delta w \qquad (8.77)$$

formula bilan hisoblanadi. Shuning uchun egri chiziqli koordinatalarda hajm elementi deb

$$dV=|J(u,v,w)|dudvdw$$

miqdorga aytiladi.

Bu natijani silindrik va sferik koordinatalarga tadbiq qilamiz.

1) (8.75) tengliklar bialn aniqlanuvchi silindrik koordinatalar uchun

$$J(r,\varphi,z)=\begin{vmatrix}\dfrac{\partial g}{\partial r} & \dfrac{\partial g}{\partial \varphi} & \dfrac{\partial g}{\partial z}\\ \dfrac{\partial h}{\partial r} & \dfrac{\partial h}{\partial \varphi} & \dfrac{\partial h}{\partial z}\\ \dfrac{\partial g}{\partial r} & \dfrac{\partial h}{\partial \varphi} & \dfrac{\partial l}{\partial z}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi & 0\\ \sin\varphi & r\cos\varphi & 0\\ 0 & 0 & 1\end{vmatrix}=r,$$

har doim $r\geq 0$ bo‘lganligi uchun silindrik koordinatalarda hajm elementi

$$dV=rdrd\varphi dz$$

tenglik bilan hisoblanadi.

2) (8.76) tengliklar bilan aniqlanuvchi sferik koordinatalar uchun

$$J(\rho,\theta,\varphi)=\begin{vmatrix}\dfrac{\partial g}{\partial \rho} & \dfrac{\partial g}{\partial \theta} & \dfrac{\partial g}{\partial \varphi}\\ \dfrac{\partial h}{\partial \rho} & \dfrac{\partial h}{\partial \theta} & \dfrac{\partial h}{\partial \varphi}\\ \dfrac{\partial g}{\partial \rho} & \dfrac{\partial h}{\partial \theta} & \dfrac{\partial l}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi\\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi\\ \cos\theta & -\rho\sin\theta & 0\end{vmatrix}=$$

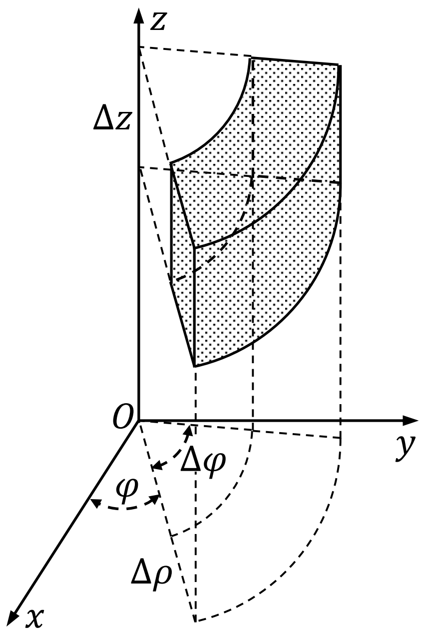
$$=\rho^2\sin\theta.$$

Bu miqdor har doim nomanfiy bo‘lganligi uchun sferik koordinatalarning hajm element

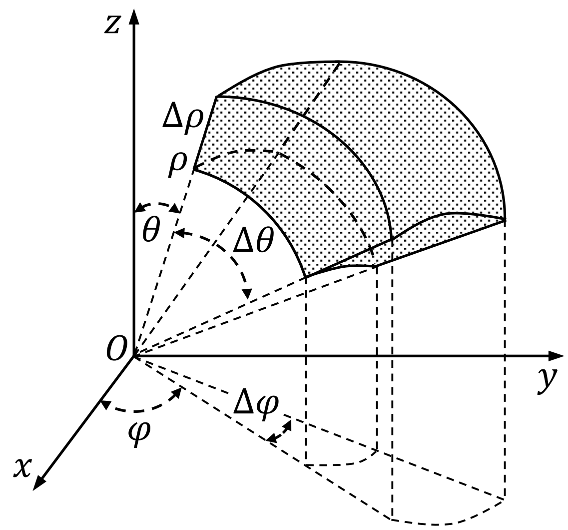
$$dV = \rho^2 \sin\theta \, d\theta d\varphi d\rho$$

bo‘ladi.

Olingan natijalarning geometrik tasviri 8.50 va 8.51-rasmlarda tasvirlangan.



8.50-rasm



8.51-rasm

Uch o‘lchovli integralda o‘zgaruvchilarni almashtirish. (8.77) formuladan Ω sohaning hajmi uchun

$$V_\Omega = \iiint\limits_{G} |J(u,v,w)| dudvdw \qquad (8.78)$$

Endi ikki o‘chovli integraldagi singri mulohaza yuritib, uch o‘lchovli integralda o‘zgaruvchilarni almashtirishning

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz =$$

$$= \iiint\limits_{G} f(g(u,v,w), h(u,v,w), l(u,v,w))|J(u,v,w)|dudvdw \qquad (8.79)$$

umumiy formulasini hosil qilamiz.

Silindrik va sferik formulalar uchun (8.79) formula

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \iiint\limits_{G} f(r\cos\varphi, r\sin\varphi, z) r dr d\varphi dz$$

$$\iiint\limits_{D} f(x,y,z)dxdydz =$$

$$= \iiint_{\Omega} f(\rho\sin\theta\cos\varphi\,, \rho\sin\theta\cos\varphi\,, \rho\cos\theta)\rho^2 \sin\theta d\rho d\varphi d\theta$$

ko‘rinishda bo‘ladi.

Uch o‘lchovli integrallarni hisoblashda ba’zan *umumlashgan sferik koordinatalardan* foydalanish qulay. Ular dekart koordinatalari bilan

$$x = a\rho\sin\theta\cos\varphi,\ y = b\rho\sin\theta\cos\varphi,\ z = c\rho\cos\theta \qquad (8.80)$$

formulalar orqali bog‘langan. Yarim o‘qlari a, b, c bo‘lgan ellipsoidning $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ tenglamasi bu koordinatalarda $\rho = 1$ ko‘rinishda bo‘lishini osongina tekshirish mumkin. Hajm elementi esa

$$dV = abc\rho^2 \sin\theta d\rho d\varphi d\theta$$

tenglik bilan hisoblanadi.

8.18-Misol. $\iiint_{\Omega} y^2 dxdydz$ integralni hisoblang, bu yerda $\Omega = \left\{(x, y, z): z \le 3, z \ge \sqrt{x^2 + y^2}\right\}$.

►D integrallash sohasi $z^2 = x^2 + y^2$ doiraviy konusning $z \ge 0$ yarim fazoda yotuvchi va $z = 3$ tekislik bilan chegaralangan ichki qismi (8.52-rasm). Bu holda silindrik koordinatalarga o‘tish qulay.$z = \sqrt{x^2 + y^2}$tenglamada$z = 3$ bo‘lganda $x^2 + y^2 = 9$ tenglamani hosil qilamiz. Shuning uchun integrallash sohasining Oxytekislikdagi proyeksiyasi markazi koordinatalar boshida va radiusi 3 bo‘lgan doiradan iborat. Demak, qutb radiusi va burchagi $0 \le r \le 3, 0 \le \varphi \le 2\pi$ kesmalarda o‘zgarar ekan. Konusning qaralayotgan qismining $z = \sqrt{x^2 + y^2}$ tenglamasi silindrik koordinatalarda $z = r$ ko‘rinishda bo‘ladi. Shuning uchun z applikata $r \le z \le 3$ kesmada o‘zgaradi. Integral ostidagi funksiya silindrik koordinatalarda $y^2 = r^2 \sin^2\varphi$ ko‘rinishda bo‘ladi. Shunday qilib, topilganlarni (8.79) formulaga qo‘yib,

$$\iiint_{\Omega} y^2 dxdydz = \int_0^{2\pi} d\varphi \int_0^3 dr \int_r^3 r^2 \sin^2\varphi\, r dz =$$

$$= \int_0^{2\pi} d\varphi \int_0^3 dr \int_r^3 r^3 \sin^2\varphi\, dz = \int_0^{2\pi} d\varphi \int_0^3 (r^3 \sin^2\varphi \cdot z|_r^3) dr =$$

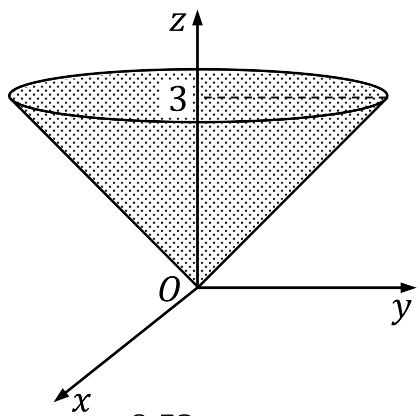
$$= \int_0^{2\pi} d\varphi \int_0^3 (r^3 \sin^2\varphi\,(3 - r) dr = \int_0^{2\pi} \sin^2\varphi \cdot \left(\frac{3r^4}{4} - \frac{r^5}{5}\right)\Bigg|_0^3 d\varphi =$$

$$= \frac{243}{20}\int_0^{2\pi} \sin^2\varphi\, d\varphi = \frac{243}{20}\int_0^{2\pi} \frac{1 - \cos 2\varphi}{2} d\varphi = \frac{243}{20}\left(\varphi - \frac{\sin 2\varphi}{2}\right)\Bigg|_0^{2\pi} =$$

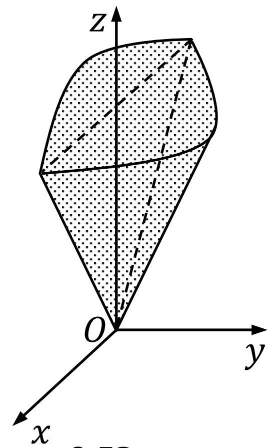
$$= \frac{243}{20} 2\pi = 12{,}15\pi$$

qiymatni hosil qilamiz.◄

8.19-Misol. $\iiint\limits_{\Omega} yzdxdydz$ integralni hisoblang, bu yerda $\Omega = \left\{(x, y, z): y \geq 0,\ z \geq \sqrt{x^2 + y^2},\ x^2 + y^2 + z^2 \leq 4\right\}$.



8.52-rasm



8.53-rasm

►Integrallash sohasi $x^2 + y^2 + z^2 \leq 4$ sharning $z^2 = x^2 + y^2$ konus bilan kesib olingan qismining $y \geq 0$ va $z \geq 0$ yarim fazolarning umumiy qismida yotuvchi bo‘lagi (8.53-rasm). Soha sharning qismi bo‘lganligi uchun sferik koordinatalarga o‘tgan qulay. Bunda θ −kenglik nuqta $z = \sqrt{x^2 + y^2}$ konusda yotganda eng katta $\theta_1 = \operatorname{arctg} 1 = \pi/4$ qiymatiga erishadi. Shuning uchun θ −kenglik $0 \leq \theta \leq \pi/4$ kesmada o‘zgaradi. Integrallash sohasining Oxy tekislikdagi proyeksiyasi markazi koordinatalar boshida va $y \geq 0$ yarim tekislikda yotuvchi yarim doira, shuning uchun $\varphi -$ uzoqlik $0 \leq \varphi \leq \pi$ kesmada o‘zgaradi. Sferaning radiusi 2 bo‘lganligi uchun ρ −radius $0 \leq \rho \leq 2$ kesmada o‘zgaradi. Integral ostidagi funksiya sferik koordinatalarda $yz = \rho^2 \sin\theta \cos\theta \sin\varphi$ ko‘rinishda bo‘ladi. Shunday qilib, topilganlarni (8.79) formulaga qo‘yib,

$$\iiint\limits_{\Omega} y^2 dxdydz = \int_0^{\pi} d\varphi \int_0^{\pi/4} d\theta \int_0^2 \rho^2 \sin\varphi \sin\theta \cos\theta \, \rho^2 \sin\theta \, d\rho =$$

$$= \int_0^{\pi} d\varphi \int_0^{\pi/4} d\theta \int_0^2 \rho^4 \sin\varphi \sin^2\theta \cos\theta \, d\rho =$$

$$= \int_0^{\pi} d\varphi \int_0^{\pi/4} \left(\sin\varphi \sin^2\theta \cos\theta\right) \frac{\rho^5}{5}\bigg|_0^2 d\theta = \frac{32}{5} \int_0^{\pi} d\varphi \int_0^{\pi/4} \sin\varphi \sin^2\theta \cos\theta \, d\theta =$$

$$=\frac{32}{5}\int_0^{\pi}\sin\varphi\frac{\sin^3\theta}{3}\Bigg|_0^{\pi/4}d\varphi=\frac{8\sqrt{2}}{15}\int_0^{\pi}\sin\varphi\,d\varphi=-\frac{8\sqrt{2}}{15}\cos\varphi\Bigg|_0^{\pi}=\frac{16\sqrt{2}}{15}$$

qidirilayotgan integralning qiymatini hosil qilamiz.◄

8.20-Misol. $\iiint\limits_{\Omega} zdxdydz$ integralni hisoblang, bu yerda integrallash sohasi

$$\Omega=\left\{(x,y,z):1\le\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}\le 4, y\le\frac{3\sqrt{3}}{2}x, y\ge\frac{3}{2}x, z\ge 0\right\}$$

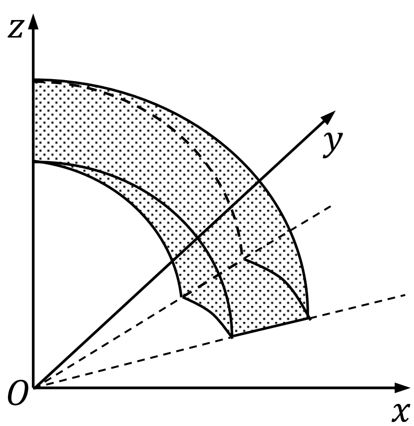
to‘plamdan iborat (8.54-rasm).

►Integrallash sohasi ellipsoidaning qismi, shuning uchun umumlashgan sferik koordinatalarga o‘tamiz: $x=2\rho\sin\theta\cos\varphi$, $y=3\rho\sin\theta\sin\varphi$, $z=4\rho\cos\theta$.

Integrallash sohasini chegaralovchi ellipsoidlarning

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1,\qquad\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=4$$

Tenglamalaridan ρ −radius uchun $1\le\rho^2\le 4$ yoki $1\le\rho\le 2$ o‘zgarish kesmasini hosil qilamiz. Sohaning Oxy tekislikdagi proyeksiyasi $x^2/4+y^2/9\le 4$ ellipsning $y=3x/2$ va $y=3\sqrt{3}x/2$ to‘g‘ri chiziqlar orasidagi qismidan iborat. Birinchi to‘g‘ri chiziqning tenglamasini yangi koordinatalarda yozamiz:



8.54-rasm

$$3\rho\sin\theta\sin\varphi=3\cdot 2\rho\sin\theta\cos\varphi/2,$$

uni soddalashtirib $\operatorname{tg}\varphi=1$ tenglikka ega bo‘lamiz. Ikkinchi to‘g‘ri chiziq uchun

$$3\rho\sin\theta\sin\varphi=3\sqrt{3}\cdot 2\rho\sin\theta\cos\varphi/2$$

va bu yerda ham soddalashtirib $\operatorname{tg}\varphi=\sqrt{3}$ tenglikni hosil qilamiz. Demak$\operatorname{arctg}1=\pi/4\le\varphi\le\operatorname{arctg}\sqrt{3}=\pi/3$, ya’ni φ −uzoqlik $\pi/4\le\varphi\le\pi/3$ kesmada o‘zgaradi. $z\ge 0$ bo‘lganligi uchun θ −kenglik $0\le\theta\le\pi/2$ kesmada o‘zgaradi. Integral ostidagi funksiya umumlashgan sferik koordinatalarda $z=4\rho\cos\theta$ ko‘rinishda bo‘ladi. Almashtirish yakobiyani moduli esa $|J(\rho,\theta,\varphi)|=2\cdot 3\cdot 4\rho^2\sin\theta=24\rho^2\sin\theta$ ga teng. Topilganlarni berilgan integralga qo‘yib

$$\iiint\limits_{\Omega} zdxdydz = \int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{0}^{\pi/2} d\theta \int\limits_{1}^{2} 4\rho\cos\theta \cdot 24\rho^2 \sin\theta\, d\rho =$$
$$= 96\int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{0}^{\pi/2} d\theta \int\limits_{1}^{2} \rho^3 \sin\theta\cos\theta\, d\rho = 96\int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{0}^{\pi/2} \sin\theta\cos\theta \left.\frac{\rho^4}{4}\right|_1^2 d\theta =$$
$$= 24\cdot 15\int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{0}^{\pi/2} \sin\theta\cos\theta\, d\theta = 24\cdot 15\int\limits_{\pi/4}^{\pi/3} \left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2} d\varphi =$$
$$= \frac{24\cdot 15}{2}\int\limits_{\pi/4}^{\pi/3} d\varphi = 12\cdot 15\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 15\pi$$

integralning qiymatini topamiz.◄

8.9. Uch o‘chovli integralning tadbiqlari

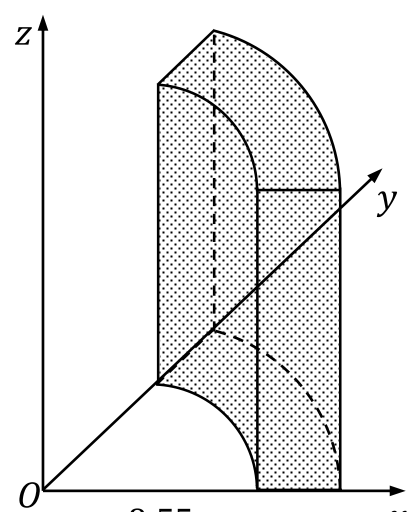
Jismning massasini hisoblash. Uch o‘lchovli integralga keltiriladigan masala sifatida biror Ω soha nuqtalarida massa taqsimlanishining $\mu(x,y,z)$ zichlik funksiyasi berilgan bo‘lsa, bu jismning massasi

$$m = \iiint\limits_{\Omega} \mu(x,y,z)dxdydz \qquad (8.81)$$

formula bilan hisoblanishini ko‘rgan edik. Biz bu yerda $\mu(x,y,z)$ funksiyani Ω sohada uzluksiz deb faraz qilamiz.

8.21-Misol. $\Omega = \left\{(x,y,z): 4 \le \frac{x^2}{9} + \frac{y^2}{4} \le 9, x \ge 0, y \ge 0, 0 \le z \le 2\right\}$ soha nuqtalarida massa $\mu(x,y,z) = (x+y)z$ zichlik funksiyaga ko‘ra taqsimlangan bo‘lsa, uning m massasini toping (8.55-rasm).

►Soha elliptik silindrning qismi, shuning uchun $x = 3r\cos\varphi$, $y = 2r\sin\varphi$, z=z deb o‘zgauvchilarni almshtirish qulay. Jism $x^2/9 + y^2/4 = 4$ va $x^2/9 + y^2/4 = 9$ elliptik silindrlar oralig‘ida joylashganligi tufayli, bu tenglamalarni yangi koordinatalarga o‘tkazsak $4 \le r^2 \le 9$ bo‘ladi, ya’ni $r-$ radius $2 \le r \le 3$ kesmada o‘zgaradi. Ω sohaning Oxy tekislikdagi proyeksiyasi $x^2/9 + y^2/4 = 4$ va $x^2/9 + y^2/4 = 9$ ellipslar oralig‘idagi sohaning birinchi chorakda yotgan



8.55-rasm

bo‘lagidan iborat. Shu sababli $\varphi -$ qutb burchagi uchun o‘zgarish kesmasi $0 \leq \varphi \leq \pi/2$ bo‘ladi. Zichlik funksiyasi yangi koordinatalarda $\mu(x,y,z) = (x+y)z = (3\cos\varphi + 2\sin\varphi)rz$, yakobiyan moduli esa $|J(r,\varphi,z)| = 2 \cdot 3r = 6r$ ko‘rinishda bo‘ladi. Topilganlarni (8.81) formulaga qo‘yib,

$$m = \iiint_{\Omega} (x+y)zdxdydz = 6\int_0^{\pi/2} d\varphi \int_2^3 dr \int_0^2 (3\cos\varphi + 2\sin\varphi)r^2 zdz =$$

$$= 6\int_0^{\pi/2} d\varphi \int_2^3 (3\cos\varphi + 2\sin\varphi)r^2 \left.\frac{z^2}{2}\right|_0^2 dr =$$

$$= 12\int_0^{\pi/2} d\varphi \int_2^3 (3\cos\varphi + 2\sin\varphi)r^2 dr =$$

$$= 12\int_0^{\pi/2} (3\cos\varphi + 2\sin\varphi)\left.\frac{r^3}{3}\right|_2^3 d\varphi =$$

$$= 12 \cdot \frac{19}{3}\int_0^{\pi/2} (3\cos\varphi + 2\sin\varphi)d\varphi =$$

$$= 12 \cdot \frac{19}{3}(3\sin\varphi - 2\cos\varphi)\Big|_0^{\pi/2} = 380$$

jismning massasini topdik. ◄

Jismning koordinata tekisliklariga nisbatan statik va inersiya momentlari. Yassi figuraning statik va inersiya momentlarini hisoblash va og‘irlik markazining koordinatalarini topish ikki o‘lchovli integral yordamida amalga oshirilgan edi. Ωjismning koordinata tekisliklariga nisbatan statik va inersiya momentlarini hisoblash uch o‘lchovli integral yordamida bajariladi.

Ω jism nuqtalaridagi zichlik $\mu(x,y,z)$ funksiya bilan berilgan bo‘lsa, bu jismning Oxy tekislikka nisbatan statik va inersiya momentlari mos ravishda

$$S_{xy} = \iiint_{\Omega} z\mu(x,y,z)dxdydz \tag{8.82}$$

$$I_{xy} = \iiint_{\Omega} z^2\mu(x,y,z)dxdydz \tag{8.83}$$

formulalar bilan hisoblanadi.

Xuddi shu singari Oyz va Oxz tekisliklarga nisbatan bu momentlar

$$S_{yz} = \iiint_{\Omega} x\mu(x,y,z)dxdydz, S_{xz} = \iiint_{\Omega} y\mu(x,y,z)dxdydz;$$

$$I_{yz} = \iiint_{\Omega} x^2\mu(x,y,z)dxdydz, I_{xz} = \iiint_{\Omega} y^2\mu(x,y,z)dxdydz$$

formulalar bilan hisoblanadi.

Jism og‘irlik markazining koordinatalarini topish. Jism og‘irlik markazining koordinatalri

$$x_c = \frac{S_{yz}}{m}, \qquad y_c = \frac{S_{xz}}{m}, \qquad z_c = \frac{S_{xy}}{m} \tag{8.84}$$

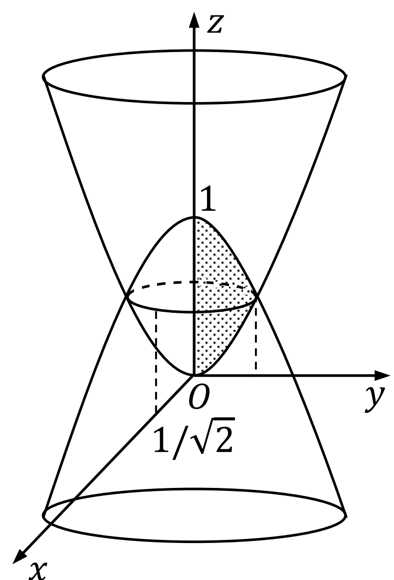
formulalarga ko‘ra topiladi. (8.81) va (8.82) formulalarni inobatga olsak bu formulalar

$$x_c = \frac{\iiint\limits_{\Omega} x\mu(x,y,z)dxdydz}{\iiint\limits_{\Omega} \mu(x,y,z)dxdydz}, \qquad y_c = \frac{\iiint\limits_{\Omega} y\mu(x,y,z)dxdydz}{\iiint\limits_{\Omega} \mu(x,y,z)dxdydz},$$

$$z_c = \frac{\iiint\limits_{\Omega} z\mu(x,y,z)dxdydz}{\iiint\limits_{\Omega} \mu(x,y,z)dxdydz}$$

ko‘rinishni oladi.

8.22-Misol. $z = x^2 + y^2$, $z = 1 - x^2 - y^2$ paraboloidlar bilan chegaralangan, birinchi oktantda yotuvchi Ω jism nuqtalaridagi zichlik nuqta koordinatalri ko‘paytmasiga proportional bo‘lsa, bu jismning koordinata tekisliklariga nisbatan statik momentlarini hisoblang va og‘irlik markazining koordinatalrini toping (8.52-rasm).



8.52-rasm

►Sohani chegaralovchi sirtlar tenglamasida x va y ozgaruvchilar kvadratlarining yig‘indisi qatnashgan, shu sababli oz’garuvchilarni silindrik koordinatalar bilan almashtirish qulay: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$. Soha yuqoridan $z = 1 - x^2 - y^2$ sirt bilan, quyidan esa $z = x^2 + y^2$ sirt bilan chegaralangan, ya’ni $x^2 + y^2 \le z \le 1 - x^2 - y^2$. Bu yerga yangi o‘zgaruvchilarni kiritsak $r^2 \le z \le 1 - r^2$ bo‘ladi. Sohaning Oxy tekislikdagi proyeksiyasi markzi koordinatalar boshida va radiusi $1/\sqrt{2}$ bo‘lgan doiraning birinchi chorakda yotgan qismidan iborat, shuning uchun $0 \le \varphi \le \pi/2$ va $0 \le r \le 1/\sqrt{2}$ bo‘ladi. Shartga ko‘ra zichlik funksiyasi $\mu(x,y,z) = kxyz$ va uning yangi koordinatalrda $kxyz = kr\cos\varphi\, r\sin\varphi\, z = kr^2\cos\varphi\sin\varphi\, z$ ko‘rinishda bo‘ladi. Topilganlarni (8.81) formulaga qo‘yib, jismning massasini topamiz:

$$m = \iiint\limits_{\Omega} \mu(x,y,z)dxdydz = k\int\limits_0^{\pi/2} d\varphi \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^2 \cos\varphi \sin\varphi\, zrdz =$$

$$= k\int\limits_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi \cdot \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^3 zdz =$$

$$= k\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/2} \cdot \int\limits_0^{1/\sqrt{2}} r^3 \frac{1}{2}z^2\Big|_{r^2}^{1-r^2} dr = k\frac{1}{4}\int\limits_0^{1/\sqrt{2}} r^3((1-r^2)^2 - r^4)dr =$$

$$= k\frac{1}{4}\int\limits_0^{1/\sqrt{2}} (r^3 - 2r^5)dr = k\frac{1}{4}\left(\frac{r^4}{4} - \frac{r^6}{3}\right)\Bigg|_0^{1/\sqrt{2}} = k\frac{1}{192};$$

Xuddi shu singari statik va inersiya momentlarini hisoblaymiz.

$$S_{xy} = \iiint\limits_{\Omega} z\mu(x,y,z)dxdydz = k\int\limits_0^{\pi/2} d\varphi \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^2 \cos\varphi \sin\varphi\, z^2 rdz =$$

$$= k\int\limits_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi \cdot \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^3 z^2 dz =$$

$$= k\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/2} \cdot \int\limits_0^{1/\sqrt{2}} r^3 \frac{1}{3}z^3\Big|_{r^2}^{1-r^2} dr = k\frac{1}{6}\int\limits_0^{1/\sqrt{2}} r^3((1-r^2)^3 - r^6)dr =$$

$$= k\frac{1}{6}\int\limits_0^{1/\sqrt{2}} (r^3 - 3r^5 + 3r^7 - 2r^9)dr =$$

$$= k\frac{1}{6}\left(\frac{r^4}{4} - \frac{r^6}{2} + \frac{3r^8}{8} - \frac{r^{10}}{5}\right)\Bigg|_0^{1/\sqrt{2}} = k\frac{11}{3840};$$

$$S_{yz} = \iiint\limits_{\Omega} x\mu(x,y,z)dxdydz =$$

$$= \int\limits_0^{\pi/2} d\varphi \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^2 \cos\varphi \sin\varphi\, r\cos\varphi\, zrdz =$$

$$= k\int\limits_0^{\pi/2} \cos^2\varphi \sin\varphi\, d\varphi \cdot \int\limits_0^{1/\sqrt{2}} dr \int\limits_{r^2}^{1-r^2} r^4 zdz =$$

$$= -k\frac{1}{3}\cos^3\varphi\Big|_0^{\pi/2} \cdot \int\limits_0^{1/\sqrt{2}} r^4 \frac{1}{2}z^2\Big|_{r^2}^{1-r^2} dr = k\frac{1}{6}\int\limits_0^{1/\sqrt{2}} r^4\left((1-r^2)^2 - r^4\right)dr =$$

$$= k\frac{1}{6}\int_0^{1/\sqrt{2}}(r^4 - 2r^6)dr = k\frac{1}{6}\left(\frac{r^5}{5} - \frac{2r^7}{7}\right)\Bigg|_0^{1/\sqrt{2}} =$$

$$= k\frac{1}{6}\cdot\frac{1}{70\sqrt{2}} = k\frac{1}{420\sqrt{2}};$$

$$S_{xz} = \iiint\limits_{\Omega} y\mu(x,y,z)dxdydz =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{1/\sqrt{2}} dr \int_{r^2}^{1-r^2} r^2\cos\varphi\sin\varphi\, r\sin\varphi\, zrdz =$$

$$= k\int_0^{\pi/2} \sin^2\varphi\cos\varphi\, d\varphi \cdot \int_0^{1/\sqrt{2}} dr \int_{r^2}^{1-r^2} r^4 zdz =$$

$$= k\frac{1}{3}\sin^3\varphi\Big|_0^{\pi/2} \cdot \int_0^{1/\sqrt{2}} r^4\frac{1}{2}z^2\Big|_{r^2}^{1-r^2} dr = k\frac{1}{6}\cdot\frac{1}{70\sqrt{2}} = k\frac{1}{420\sqrt{2}}.$$

Topilganlarni (8.84) formulalarga qo‘yib, jism og‘irlik markazining koordinatalarini topamiz:

$$x_c = \frac{S_{yz}}{m} = \frac{k\frac{1}{420\sqrt{2}}}{k\frac{1}{192}} = \frac{16}{35\sqrt{2}};\quad y_c = \frac{S_{xz}}{m} = \frac{k\frac{1}{420\sqrt{2}}}{k\frac{1}{192}} = \frac{16}{35\sqrt{2}};$$

$$z_c = \frac{S_{xy}}{m} = \frac{k\frac{11}{3840}}{k\frac{1}{192}} = \frac{33}{60}.$$

Jismning shakli va zichlik funksiyasi x va y o‘zgaruvchilarga nisbatan simmetrikligi tufayli og‘irlik markazining abssissasi va ordinatasi teng.◄

Nazorat savollari

1. Ikki o‘lchovli integral tushunchasiga keltriladigan masala: silindrik jism hajmini hisoblash haqida masala.
2. Ikki o‘zgaruvchili funksiyaning biror soha bo‘yicha integral yig‘indisi qanday tuziladi.
3. Ikki o‘lchovli integralning ta’rifini bayon qiling.
4. Ikki o‘lchovli funksiyaning quyi va yuqori Darbu yig‘indilari qanday tuziladi.
5. Ikki o‘lchovli integralning xossalarini bayon qiling.
6. Ikki o‘zgaruvchili funksiyaning biror sohadagi o‘rta qimatiga ta’rif bering.

7. Ikki o‘lchovli integralni to‘g‘ri tortburchak soha bo‘yicha hisoblash formulasini yozing.
8. Ox yoki Oy o‘q yonalishi bo‘yicha to‘g‘ri soha ta’rifini keltiring.
9. Ikki o‘lchovli integralni ixtiyoriy soha bo‘yicha hisoblash formulasini yozing.
10. Tekislikdagi nuqtaning egri chiziqli koordinatalariga ta’rif bering.
11. $x(u,v)$ va $y(u,v)$ funksiyalarning yakobianiga ta’rif bering.
8. Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish formulasini yozing.
13. Qutb koordinatalarida ikki o‘lchovli integralni hisoblash formulasini yozing.
14. Yassi figuraning yuzini ikki o‘lchovli integral yordamida hisoblash formulasini yozing.
15. Fazoviy sirt yuzini ikki o‘lchovli integral yordamida hisoblash formulasini yozing.
16. Puasson integrali qanday hisoblanadi?
17. Yassi figuraning hajmini hisoblash formulasini yozing.
18. Yassi figuraning koordinata o‘qlariga nisbatan statik momentlari qanday topiladi?
19. Yassi figuraning koordinata o‘qlariga nisbatan inersiya momentlari qanday topiladi?
20. Yassi figura og‘irlik markazining koordinatalari qanday topiladi?

Mashqlar.

Quyida berilgan Dsoha bo‘yicha olingan $\iint\limits_D f(x,y)dxdy$ ikki o‘lchovli integralni takroriy integral ko‘rinishida yozing:

1. $D=\{(x,y): x\leq -|y|, x^2+y^2\leq -2x\}$. **2.** $D=\{(x,y): y\leq -|x|, x^2+y^2\leq -2y\}$. **3.** $D=\{(x,y): y\leq 0, y\geq x^2-9\}$. **4.** $D=\{(x,y): y\geq x^2, y\leq 16-x^2\}$. **5.** D soha uchlari $A(2{,}0), B(5{,}0), C(5,-6)$ nuqtalarda bo‘lgan uchburchakning ichki qismi.

6. D soha uchlari $A(0{,}0), B(3{,}2), C(4{,}1), D(3,-2)$ nuqtalarda bo‘lgan to‘rt burchakning ichki qismi.

Takroriy integrallarda integrallash tartibini o‘zgartiring:

7. $\int\limits_0^4 dx \int\limits_0^{2\sqrt{x}} f(x,y)dy + \int\limits_4^6 dx \int\limits_0^{(x-6)^2} f(x,y)dy$

8. $\int_0^2 dx \int_0^{2x^2} f(x,y)dy + \int_2^4 dx \int_0^{-(x-4)^3} f(x,y)dy$

Takroriy integrallarni hisoblang:

9. $\int_2^3 dy \int_0^{\ln y} e^x dx$. **10.** $\int_0^{\sqrt{\pi}} dx \int_x^{\sqrt{\pi}} \sin y^2\, dy$. **11.** $\int_0^1 dx \int_0^1 ye^{xy} dy$.

Ikki o‘lchovli integrallarni hisoblang:

12. $\iint_D \sqrt{xy-y^2}dxdy$, bu yerda D integrallash sohasi uchlari $A(1,1)$, $B(5,1)$, $C(10,2)$, $D(2,2)$ nuqtalarda bo‘lgan trapetsiyadan iborat.

13. $\iint_D ydxdy$, bu yerda D integrallash sohasi uchlari $O(0,0)$, $A(1,1)$, $B(0,1)$ nuqtalarda bo‘lgan uchburchakdan iborat.

14. $\iint_D e^{x^2+y}dxdy$, bu yerda $D = \{(x,y): 1 \leq x \leq 2, y \leq \ln 2x\,, y \geq \ln x\}$.

15. $\iint_D (x+y^2)dxdy$, bu yerda $D = \{(x,y): |x| + |y| \leq 1\}$.

16. $\iint_D xy^2dxdy$, bu yerda $D = \{(x,y): x \leq 3, y \leq x, xy \geq 1\}$.

17. $\iint_D \sqrt{4-x^2-y^2}dxdy$, bu yerda D markazi koordinatalar boshida va radiusi 2 bo‘lgan doiraning birinchi chorakdagi bo‘lagi.

18. $\iint_D (x^2+y^2)dxdy$, bu yerda D ushbu $x^2+y^2=4x$, $x^2+y^2=8x$, $y=0, (y>0)$ chiziqlar bilan chegaralangan soha.

19. $y^2 = 4x+4$ va $x+y=2$ egri chiziqlar bilan chegaralangan figuraning yuzini toping.

20. $x^2+y^2=4$ silindr sirtidan $y^2=2(2-x)$ silindr bilan kesib olingan qismimning yuzini toping.

21. $x^2+z^2=y^2$ konus sirtidan $y^2=2x$ silindr bilan kesib olingan qismimning yuzini toping.

22. Katetlari $OB=3$ va $OA=4$ bo‘lgan to‘g‘ri burchakli uchburchak shakldagi yassi plastinkaning nuqtadagi zichligi bu nuqtadan OA katetgacha bo‘lgan masofaga teng bo‘lsa, bu plastinkaning massasini toping.

Quyida berilgan D soha bo‘yicha olingan $\iiint_D f(x,y,z)dxdy$ uch o‘lchovli integralni takroriy integral ko‘rinishida yozing:

23. $D = \{(x,y,z): x \geq 0, y \geq 0, z \geq 0, 2x+y \leq 2, z \leq x^2+y^2+3\}$.

24. $D = \{(x,y,z): x \geq 0, y \geq 0, z \geq 0, 2x+3y \leq 6, x+y+z-5 \leq 0\}$

Takroriy integrallarni hisoblang:

25. $\int_0^1 dx \int_x^{2x} dy \int_0^{x+y} (x+y+z)dz$. **26.** $\int_{-1}^1 dy \int_0^{y^2} dx \int_0^{x^2+y^2} (1+z)dz$.

Uch o‘lchovli integrallarni hisoblang:

27. $\iiint_D \sqrt{x}dxdydz$, bu yerda $D = \{(x,y,z): x^2+y^2 \le 2x, 0 \le z \le 2\}$.

28. $\iiint_D z\sqrt{x^2+z^2}dxdydz$, bu yerda $D = \{(x,y,z): x^2+z^2 \le y^2, 0 \le y \le 2\}$.

Quyidagi uch o‘lchovli integrallarni silindrik yoki sferik almashtirishlar yordamida hisoblang:

29. $\iiint_D x^2dxdydz$, bu yerda $D = \{(x,y,z): 1 \le x^2+y^2+z^2 \le 4, -z \le y \le z\}$.

30. $\iiint_D ydxdydz$, bu yerda $D = \{(x,y,z): x \ge 0, y \ge 0, x^2+y^2+z^2 \le 4, x^2+y^2+z^2 \le 3z\}$.

31. $x^2+y^2+z^2 \le 4$, $z \ge 0$ yarim sharning nuqtadagi zichligi bu nuqtadan koordinata boshigacha bo‘lgan masofaga proporsional bo‘lsa, yarim shar og‘irlik markazi koordinatalarini toping.

Javoblar.

1. $\int_{-2}^{-1} dx \int_{-\sqrt{-2x-x^2}}^{\sqrt{-2x-x^2}} f(x,y)dy + \int_{-1}^{0} dx \int_{x}^{-x} f(x,y)dy =$

$= \int_{-1}^{0} dy \int_{-1-\sqrt{1-y^2}}^{y} f(x,y)dx + \int_{0}^{1} dy \int_{-1-\sqrt{1-y^2}}^{-y} f(x,y)dx$.

2. $\int_{-1}^{0} dx \int_{-1-\sqrt{1-x^2}}^{x} f(x,y)dx + \int_{0}^{1} dy \int_{-1-\sqrt{1-x^2}}^{-x} f(x,y)dx =$

$= \int_{-2}^{-1} dy \int_{-\sqrt{-2y-y^2}}^{\sqrt{-2y-y^2}} f(x,y)dy + \int_{-1}^{0} dx \int_{y}^{-y} f(x,y)dy$. **3.** $\int_{-3}^{3} dx \int_{x^2-9}^{0} f(x,y)dy =$

$= \int_{-9}^{0} dy \int_{-\sqrt{y+9}}^{\sqrt{y+9}} f(x,y)dy$. **4.** $\int_{-2\sqrt{2}}^{2\sqrt{2}} dx \int_{x^2}^{16-x^2} f(x,y)dy = \int_0^8 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx +$

$+ \int_8^{16} dy \int_{-\sqrt{16-y}}^{\sqrt{16-y}} f(x,y)dx$. **5.** $\int_2^5 dx \int_{-2x+4}^{0} f(x,y)dy = \int_{-6}^{0} dy \int_{(4-y)/2}^{5} f(x,y)dy$.

6. $\int_0^3 dx \int_{-2x/3}^{2x/3} f(x,y)dy + \int_3^4 dx \int_{3x-11}^{-x+5} f(x,y)dy = \int_{-2}^{0} dy \int_{-3y/2}^{(y+11)/3} f(x,y)dx +$
$+\int_0^1 dy \int_{3y/2}^{(y+11)/3} f(x,y)dx + \int_1^2 dy \int_{3y/2}^{-y+5} f(x,y)dx$. **7.** $\int_0^4 dy \int_{y^2/4}^{6-\sqrt{y}} f(x,y)dx$.
8. $\int_0^8 dy \int_{\sqrt{y/2}}^{4-\sqrt[3]{y}} f(x,y)dx$. **9.** 1,5. **10.** 1. **11.** $e-2$. **12.** $112/9$. **13.** $1/3$.
14. $\int_1^2 dx \int_{\ln x}^{\ln 2x} e^{x^2+y}dy = (e^4-e)/2$.
15. $\int_{-1}^{0} dx \int_{-x-1}^{x+1} (x+y^2)dy + \int_0^1 dx \int_{x-1}^{-x+1} (x+y^2)dy = 1/3$.
16. $\int_1^3 dx \int_{1/x}^{x} xy^2dy = 716/45$. **17.** $4\pi/3$. **18.** 180π. **19.** $64/3$. **20.** $32\sqrt{2}$. **21.** $2\sqrt{2}\pi$. **22.** 6. **23.** $\int_0^1 dx \int_0^{2-2x} dy \int_0^{x^2+y^2+3} f(x,y,z)dz$.
24. $\int_0^3 dx \int_0^{(6-2x)/3} dy \int_0^{5-x-y} f(x,y,z)dz$. **25.** $19/8$. **26.** $7592/10395$.
27. $\int_0^2 dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dy \int_0^2 \sqrt{x}dz = 64\sqrt{2}/15$.
28. $\int_0^2 dy \int_{-y}^{y} dx \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} \sqrt{x^2+y^2}z dz = 0$. **29.** $31\pi/15$. **30.** 13,5. **31.** $C(0,0,4/5)$.

IX BOB. EGRI CHIZIQLI INTEGRAL VA SIRT INTEGRALLARI

Integrallash sohasi biror egri chiziq bo‘lgan aniq integralning umumlashtirilishi egri chiziqli integral deb ataladi.

9.1. I tur egri chiziqli integral

Egri chiq yoyi uzunligi tushunchasi va tenglamalari turli xil ko‘rinishda berilgan egri chiziqlar uchun bu uzunliklarni hisoblash formulalaridan foydalanib, to‘g‘ri chiziq kesmasi bo‘ylab kiritilgan aniq integral singari egri chiziq bo‘ylab integral tushunchasini kiritish mumkin.

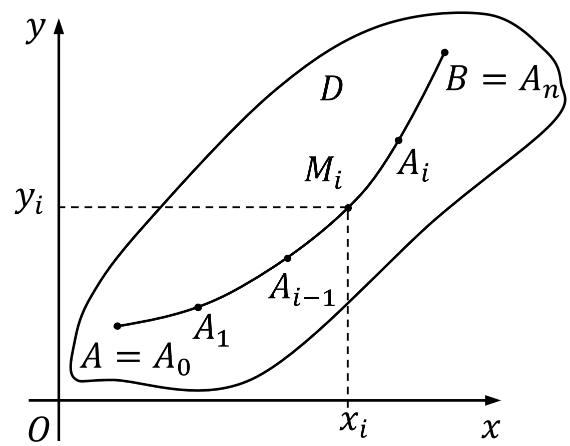
AB (yoki L) egri chiziq

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} \quad \alpha \le t \le \beta$$

paramaetrik tenglama bilan berilgan bo‘lsin. Agar $x(t)$ va $y(t)$ funksiyalar $[\alpha, \beta]$ kesmada uzluksiz $x'(t), y'(t)$ hosilalarga ega va bu hosilalar ana shu kesma nuqtalarida

$$[x'(t)]^2 + [y'(t)]^2 > 0$$

tengsizlikni qanoatlantirsa, AB egri *chiziq silliq egri chiziq* deb ataladi. Agar $[\alpha, \beta]$ kesmaning chekli sondagi nuqtalarida bu hosilalar mavjud bo‘lmasa yoki ikkala hosila bir vaqtda nolga aylansa, egri chiziq *bo‘lakli silliq* deb ataladi. $f(x, y)$ funksiya AB egri chiziq yotgan biror D sohada aniqlangan va uzluksiz bo‘lsin.



9.1-rasm

AB egri chiziqni $A_0 = A$, A_1, $\dots, A_n = B$ nuqtalar yordamida n ta elementar yoylarga ajratamiz. Bu bo‘linishni $T = \{A_0, A_1, \dots, A_n\}$ orqali, $A_{i-1}A_i$ yoy uzunligini Δs_i orqali va bu uzunliklarning eng kattasini $\lambda = \lambda(T)$ orqali belgilaymiz. Har bir yoyda ixtiyoriy ravishda $M_i(x_i, y_i)$ nuqtani tanlaymiz (9.1-rasm).

Quyidagi yig‘indini tuzamiz:

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \tag{9.1}$$

va uni $f(x, y)$ funksiyaning AB egri chiziq bo‘ylab integral yig‘indisi deb ataymiz.

9.1-Ta'rif. Agar eng katta λ yoy uzunligi nolga intilganda: $\lambda \to 0$ (u holda $n \to \infty$ bo'ladi), (9.1) integral yig'indilar chekli limitga intilsa va bu limit AB egri chiziqni T bo'lishlariga va $M_i(x_i, y_i)$ nuqtalarning tanlanishiga bog'liq bo'lmasa, uni biz $f(x,y)$ funksiyaning *AB egri chiziq uzunligi bo'ylab integrali* yoki *I tur egri chiziqli integral* deb ataymiz va uni $\int\limits_{AB} f(x,y)ds$ (yoki $\int\limits_{L} f(x,y)ds$) orqali belgilaymiz.

Shunday qilib, ta'rifga ko'ra

$$\int\limits_{AB} f(x,y)ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i. \tag{9.2}$$

Bu holda $f(x,y)$ funksiya AB egri chiziq bo'ylab integrallanuvchi, AB egri chiziq integrallash konturi, $A -$integrallashning boshlang'ich, $B -$esa oxirgi nuqtasi deb ataladi.

(9.1) integral yig'indi AB egri chiziqdagi yo'nalishga bog'liq bo'lmaganligi tufayli I tur egri chiziqli integral ham egri chiziqdagi yo'nalishga bog'liq bo'lmaydi. AB egri chiziq yopiq bo'lmasin, BA egri chiziq ham xuddi shu egri chiziq deb tushuniladi, faqat yo'nalish B nuqtadan A nuqtaga qarab yo'nalgan deb hisoblanadi. U holda

$$\int\limits_{AB} f(x,y)ds = \int\limits_{BA} f(x,y)ds \tag{9.3}$$

tenglik o'rinli bo'ladi.

Fazoviy egri chiziq bo'ylab integral ham xuddi shu singari kiritiladi. AB fazoviy egri chiziq yotgan Ω sohada $f(x,y,z)$ funksiya aniqlangan bo'lsin. AB egri chiziqni $A_0 = A,\ \ A_1, \dots, A_n = B$ nuqtalar yordamida $A_{i-1}A_i$ elementar yoylarga ajratuvchi $T = \{A_0, A_1, \dots, A_n\}$ bo'linishni bajaramiz. Har bir $A_{i-1}A_i$, $(i = 1,2, \dots n)$ yoyda $M_i(x_i, y_i, z_i)$ nuqtani tanlab,

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

integral yig'indini tuzamiz va $\lambda(T) \to 0$ bo'lganda limitga o'tamiz, natijada fazoviy AB egri chiziq bo'ylab integral qiymatini hosil qilamiz:

$$\int\limits_{AB} f(x,y,z)ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i, \tag{9.4}$$

bu yerda ham $A_{i-1}A_i$ yoy uzunligini Δs_i orqali va bu uzunliklarning eng kattasini $\lambda = \lambda(T)$ orqali belgilangan.

9.1-Misol. Biror L silliq chiziq bo‘ylab $f(M)$ o‘zgaruvchan chiziqli zichlikka ega bo‘lgan massa taqsimlangan bo‘lsin. L chiziqning m massasini topilsin.

►L chiziqni n ta $M_{i-1}M_i (i = 1,2, \dots , n)$ qismlarga ajratamiz va har bir qismda zichlik o‘zgarmas va biror nuqtadagi, masalan M_i nuqtadagi $f(M_i)$ zichlikka teng deb faraz qilib, har bir qismning massasini hisoblaymiz. U holda

$$\sum_{i=1}^{n} f(M_i)\Delta s_i$$

yig‘indi m massaning qiymatiga taqriban teng bo‘ladi, bu yerda Δs_i ko‘paytuvchi $i-$ qismning uzunligi. L egri chiziqni bo‘lishlar qanchalik mayda bo‘lsa, xatolik ham shunchalik kichik bo‘lishi ravshan. $\lambda \to 0$ ($\lambda = \max\limits_{i} \Delta s_i$) bo‘lganda limitga o‘tsak, L chiziqning aniq m massasini hosil qilamiz, ya’ni

$$m = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(M_i)\Delta s_i.$$

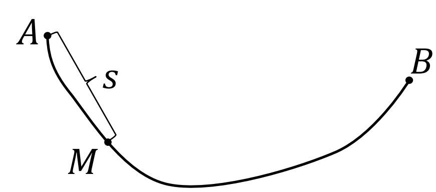
Biroq o‘ng tomondagi limit I tur egri chiziqli integraldan iborat. Demak

$$m = \int_L f(x, y)ds. ◄$$

9.2. I tur egri chiziqli integralning mavjudligi

AB egri chiziqda parametr sifatida A nuqtadan hisoblanadigan s yoy uzunligini olamiz (9.2-rasm), ya’ni egri chiziqdagi M nuqtaning holati A nuqtadan M nuqtagacha bo‘lgan $s = \overset{\frown}{AM}$ yoy uzunligi bilan aniqlanadi. U holda AB egri chiziq

$$x = x(s), y = y(s) \quad (0 \le s \le s_{AB})$$



9.2-rasm

parametrik tenglama bilan ifodalanadi, egri chiziq nuqtalarida aniqlangan $f(x, y)$ funksiya esa s argumentning murakkab $f(x(s), y(s))$ funksiyasiga o‘tadi. s_i $(i = 1,2, \dots , n)$ orqali s parametrning egri chiziqdagi A_i bo‘linish nuqtasiga mos keluvchi qiymatini belgilaymiz. U holda $\Delta s_i = s_i - s_{i-1}$ ekanligi ravshan. $\bar{s}_i$ orqali esa s parametrning M_i nuqtani aniqlovchi qiymatini belgilaymiz. Bu belgilash uchun $s_{i-1} \le \bar{s}_i \le s_i$ tengsizlik o‘rinli bo‘ladi. Kiritilgan belgilashlardan foydalanib, (9.1) integral yig‘indini

$$\sum_{i=1}^{n} f(x(\bar{s}_i), y(\bar{s}_i))\Delta s_i \tag{9.5}$$

ko‘rinishda yozib olamiz.

Bu yig‘indi bir vaqtning o‘zida $f(x(s), y(s))$ funksiya aniq integralining integral yig‘indisidan iborat va

$$\int_0^{s_{AB}} f(x(s), y(s))ds = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(x(\bar{s}_i), y(\bar{s}_i))\Delta s_i \tag{9.6}$$

tenglik o‘rinli bo‘ladi, bu yerda har doimgidek $\lambda = \max_i \Delta s_i$. (9.2) va (9.6) tengliklar o‘ng tomonlaridagi limitlar ostidagi yig‘indilar teng, shuning uchun bu limitlar ham va demak chap tomonlari ham teng bo‘ladi. Shunday qilib,

$$\int_{AB} f(x,y)ds = \int_0^{s_{AB}} f(x(s), y(s))ds \tag{9.7}$$

va ulardan birining mavjudaligi ikkinchisining ham mavjudligini keltirib chiqaradi. Shuning uchun o‘ng tomondagi aniq integralning mavjudlik sharti chap tomondagi I tur egri chiziqli integral uchun ham mavjudlik shartini beradi.

9.1-Teorema. Agar AB egri chiziq bo‘lakli silliq va $f(x,y)$ funksiya AB egri chiziqda uzluksiz bo‘lsa,

$$\int_{AB} f(x,y)ds$$

I tur egri chiziqli integral mavjud.

► AB egri chiziqni aniqlovchi $x(s)$ va $y(s)$ parametrik funksiyalar $[0, s_{AB}]$ kesmada uzluksiz bo‘lganligi uchun, $f(x,y)$ funksiya AB egri chiziqda uzluksiz bo‘lsa, murakkab funksiyaning uzluksizligi haqidagi teoremaga ko‘ra $f(x(s), y(s))$ funksiya $[0, s_{AB}]$ kesmada uzluksiz bo‘ladi. U holda 9.3-Teoremaga ko‘ra (9.7) tenglikning o‘ng tomonidagi aniq integral mavjud, bu esa $\int_{AB} f(x,y)ds$ egri chiziqli integralni mavjudligini ta’minlaydi.◄

9.3. I tur egri chiziqli integralning xossalari

Quyida I tur egri chiziqli integralning asosiy xossalarini keltiramiz. Ularning isbotini aniq integrallardagi singari (9.1) integral yig‘indining xususiyatlaridan osongina keltirib chiqarish mumkin. Ularni tekshirib ko‘rishni kitobxonga havola qilamiz.

1. I tur egri chiziqli integral integrallash yo‘lining yo‘nalishiga bog‘liq emas, ya’ni

$$\int\limits_{AB} f(x,y)ds = \int\limits_{BA} f(x,y)ds.$$

2. Agar $f(x,y)$ funksiya AB egri chiziq bo‘ylab integrallanuvchi bo‘lsa, $c \cdot f(x,y)$ funksiya ham integrallanuvchi bo‘ladi va

$$\int\limits_{AB} c \cdot f(x,y)ds = c \cdot \int\limits_{AB} f(x,y)ds,\ c = \text{const}$$

tenglik o‘rinli bo‘ladi.

3. Agar $f(x,y)$ va $g(x,y)$ funksiyalar AB egri chiziq bo‘ylab integrallanuvchi bo‘lsa, $f(x,y) \pm g(x,y)$ funksiyalar ham integrallanuvchi bo‘ladi va

$$\int\limits_{AB} (f(x,y) \pm g(x,y))ds = \int\limits_{AB} f(x,y)ds \pm \int\limits_{AB} g(x,y)ds$$

tengliklar o‘rinli bo‘ladi.

4. Agar AB egri chiziqni $C \in AB$ nuqta yordamida $AB = AC \cup CB$, AC va CB faqat bitta umumiy nuqtaga ega bo‘ladigan qilib ikki qismga ajratilsa, AB egri chiziq bo‘ylab integrallanuvchi $f(x,y)$ funksiya uchun

$$\int\limits_{AB} f(x,y)ds = \int\limits_{AC} f(x,y)ds + \int\limits_{CB} f(x,y)ds$$

tenglik o‘rinli bo‘ladi.

5. Agar AB egri chiziq bo‘ylab integrallanuvchi $f(x,y)$ va $g(x,y)$ funksiyalar AB chiziqning ixtiyoriy nuqtasida $f(x,y) \le g(x,y)$ tengsizlikni qanotlantirsa,

$$\int\limits_{AB} f(x,y)ds \le \int\limits_{AB} g(x,y)ds$$

bo‘ladi.

6. $\int\limits_{AB} ds = s_{AB}$.

7. (O‘rta qiymat haqidagi teorema) Agar $f(x,y)$ funksiya AB egri chiziqda uzluksiz bo‘lsa, u holda bu chiziqda shunday $C(x_C, y_C)$ nuqta topildiki, bu nuqta uchun $\int\limits_{AB} f(x,y)ds = f(x_C, y_C)s_{AB}$ tenglik o‘rinli bo‘ladi.

9.4. I tur egri chiziqli integralni hisoblash

Biz yuqorida I tur egri chiziqli integralni hisoblashni (9.7) formula yordamida aniq integralni hisoblashga keltirildi. Biroq bu formula amaliy jihatdan unchalik qulay emas, chunki kamdan-kam hollardagina egri chiziqning paramateri sifatida yoy uzunligi olinadi.

Bo‘lakli silliq AB egri chiziq

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} \quad \alpha \le t \le \beta \tag{9.8}$$

paramaetrik tenglama bilan berilgan bo‘lsin. U holda s yoy uzunligini egri chiziqning ixtiyoriy uchidan boshlab aniqlash mumkin, biz bu holda parametrning $t = \alpha$ qiymatiga mos keluvchi uchini boshlang‘ich nuqta sifatida qabul qilamiz. U holda t parametrning o‘sib borishiga s parametrning o‘sishi mos keladi va egri chiziq uzunligining differensiali uchun

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

formula o‘rinli bo‘ladi ((6.52) formulaga qarang). Bunda $t = \alpha$ qiymat A nuqtaga va $s = 0$ qiymatga, $t = \beta$ qiymat esa B nuqtaga va $s = s_{AB}$ qiymatga mos keladi. (9.7) tenglikdagi aniq integralda s yoy uzunligi parametridan t parametrga o‘tib, o‘zgaruvchilarni almashtirish mumkin. U holda bu tenglik

$$\int_{AB} f(x,y)ds = \int_{\alpha}^{\beta} f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt \tag{9.9}$$

ko‘rinishni oladi.

Agar yassi chiziq $y = y(x), x \in [a, b]$ funksiyaning grafigidan iborat bo‘lsa, (9.9) formula

$$\int_{AB} f(x,y)ds = \int_{a}^{b} f(x, y(x))\sqrt{1 + (y'(x))^2}dx \tag{9.10}$$

ko‘rinishda bo‘ladi. Agar egri chiziq $x = x(y),\ y \in [c, d]$ funksiya bilan berilgan bo‘lsa

$$\int_{AB} f(x,y)ds = \int_{c}^{d} f(x(y), y)\sqrt{1 + (x'(y))^2}dy \tag{9.11}$$

formulaga ega bo‘lamiz.

AB egri chiziq qutb koordinatalar sistemasida $r = r(\varphi), \varphi \in [\varphi_1, \varphi_2]$ tenglama bilan berilgan bo‘lsin. U holda dekart va qutb koordinatalarini bog‘lovchi $x = r\cos\varphi$ va $y = r\sin\varphi$ formulalarni, hamda qutb koordinatalaridagi yoy uzunligi differensialining

$$ds = \sqrt{r^2(\varphi) + (r'(\varphi))^2}d\varphi$$

ifodasini inobatga olsak

$$\int_{AB} f(x,y)ds = \int_{\varphi_1}^{\varphi_2} f(r\cos\varphi, r\sin\varphi)\sqrt{r^2(\varphi) + (r'(\varphi))^2}d\varphi \tag{9.12}$$

formulani hosil qilamiz.

Fazoviy silliq AB egri chiziq

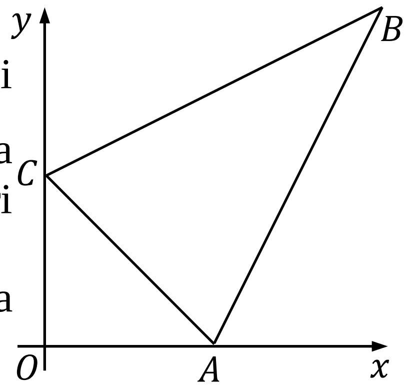
$$\begin{cases} x = x(t), \\ y = y(t), \quad \alpha \le t \le \beta \\ z = z(t) \end{cases}$$

parametrik tenglama bilan berilgan bo‘lsa, bu egri chiziqda aniqlangan va uzluksiz $f(x,y,z)$ funksiyaning (9.4) I tur egri chiziqli integrali

$$\int_{AB} f(x,y,z)ds =$$

$$= \int_{\alpha}^{\beta} f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt \qquad (9.13)$$

formulaga ko‘ra hisoblanadi.



9.3-rasm

9.2-Misol. $\int_L (x+y)ds$ egri chiziqli integralni hisoblang, bu yerda L −uchlari $A(1,0)$, $B(2,2)$ va $C(0,1)$ nuqtalarda bo‘lgan uchburchak konturi (9.3-rasm).

►I tur egri chiziqli integralning 4-xossasiga ko‘ra

$$\int_L (x+y)ds = \int_{AB} (x+y)ds + \int_{BC} (x+y)ds +$$
$$+ \int_{CA} (x+y)ds$$

bo‘ladi. Har bir integralni alohida hisoblaymiz. Ikki nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasiga ko‘ra kesmalar $AB: y = 2x - 2$ $(1 \le x \le 2)$, $BC: y = x/2 + 1$ $(0 \le x \le 2)$ va $CA: y = 1 - x$ $(0 \le x \le 1)$ tenglamalarga ega bo‘ladi. Ularning hosilalari esa mos ravishda $AB: y' = 2$, $BC: y' = 1/2$ va $CA: y' = -1$ ga teng. Topilganlarni (9.10) formulaga qo‘yib integrallarni hisoblaymiz:

$$\int_{AB} (x+y)ds = \int_1^2 (x + 2x - 2)\sqrt{1+2^2}dx = \sqrt{5}\int_1^2 (3x-2)dx =$$

$$= \sqrt{5}\left(\frac{3x^2}{2} - 2x\right)\Bigg|_1^2 = \frac{3\sqrt{5}}{2};$$

$$\int_{BC} (x+y)ds = \int_0^2 (x + x/2 + 1)\sqrt{1+1/2^2}dx = \frac{\sqrt{5}}{2}\int_0^2 (3x/2+1)dx =$$

$$= \frac{\sqrt{5}}{2}\left(\frac{3x^2}{4} + x\right)\Bigg|_0^2 = \frac{5\sqrt{5}}{2};$$

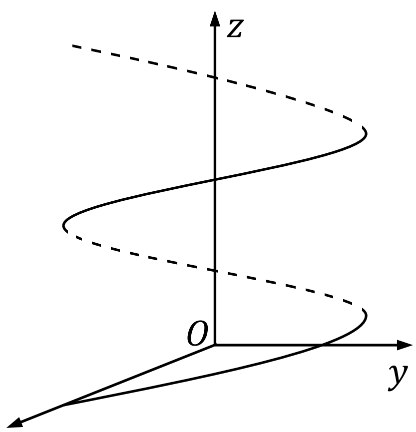
$$\int_{CA}(x+y)ds=\int_0^1(x+1-x)\sqrt{1+(-1)^2}dx=\sqrt{2}\int_0^1 dx=\sqrt{2}.$$

Demak,

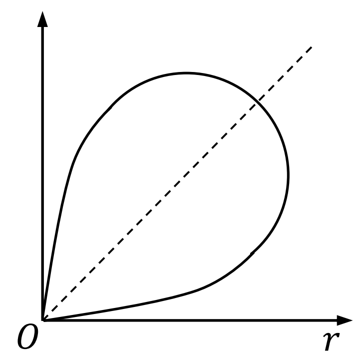
$$\int_L(x+y)ds=\frac{3\sqrt{5}}{2}+\frac{5\sqrt{5}}{2}+\sqrt{2}=4\sqrt{5}+\sqrt{2}. \qquad ◀$$

9.3-Misol. $\int_L \frac{z^2}{x^2+y^2}ds$ egri chiziqli integralni hisoblang, bu yerda $L-$vint chizig‘i deb ataluvchi burama chiziqning birinchi buramasi (9.4-rasm):

$$\begin{cases} x=a\cos t\,, \\ y=a\sin t\,, \quad 0\le t\le 2\pi. \\ z=ht, \end{cases}$$



9.4-rasm



9.5-rasm

►Hosilalarni topamiz: $x'(t)=-a\sin t$, $y'(t)=a\cos t$, $z'(t)=h.$ U holda (9.13) formulaga ko‘ra

$$\int_L \frac{z^2}{x^2+y^2}ds=\int_0^{2\pi}\frac{h^2t^2}{a^2(\cos^2 t+\sin^2 t)}\sqrt{a^2(\sin^2 t+\cos^2 t)+h^2}dt=$$

$$=\int_0^{2\pi}\frac{h^2t^2}{a^2(\cos^2 t+\sin^2 t)}\sqrt{a^2(\sin^2 t+\cos^2 t)+h^2}dt=$$

$$=\frac{h^2\sqrt{a^2+h^2}}{a^2}\int_0^{2\pi}t^2dt=\frac{h^2\sqrt{a^2+h^2}}{a^2}\frac{t^3}{3}\Big|_0^{2\pi}=\frac{8\pi^3h^2\sqrt{a^2+h^2}}{3a^2}. \qquad ◀$$

9.4-Misol. $\int_L(x+y)ds$ egri chiziqli integralni hisoblang, bu yerda $L-$ integrallash chizig‘i $r=\sqrt{\sin 2\varphi}$ limniskata bargining birinchi chorakdagi bo‘lagi (9.5-rasm).

►Hosilani topamiz: $r'(\varphi) = \cos 2\varphi / \sqrt{\sin 2\varphi}$, hamda $0 \leq \varphi \leq \pi/2$ ekanligini inobatga olib, (9.12) formulaga ko'ra

$$\int_L (x+y)ds = \int_0^{\pi/2} (r\cos\varphi + r\sin\varphi)\sqrt{\sin 2\varphi + \frac{\cos^2 2\varphi}{\sin 2\varphi}}d\varphi =$$

$$= \int_0^{\pi/2} r(\cos\varphi + \sin\varphi)\frac{1}{\sqrt{\sin 2\varphi}}d\varphi = \int_0^{\pi/2} r(\cos\varphi + \sin\varphi)\frac{1}{r}d\varphi =$$

$$= \int_0^{\pi/2} (\cos\varphi + \sin\varphi)d\varphi = 2.◄$$

9.5. I tur egri chiziqli integralning tadbiqlari

Moddiy chiziqning massasi. AB egri chiziqda aniqlangan $\mu(M)$ funksiya egri chiziq bo'ylab taqsimlangan massaning zichligini bersin. AB egri chiziqni katta $n \in \mathbb{N}$ sondagi $A_{i-1}A_i, i = 1,2, \dots, n$ elementar yoylarga bo'lganda, har bir $A_{i-1}A_i$ yoyning barcha nuqtalarida zichlik o'zgarmas va bu yoyning ixtiyoriy $M_i(x_i, y_i)$ nuqtasidagi $\mu(M_i)$ zichlikka taqriban teng deb olish mumkin. Δs_i orqali $A_{i-1}A_i$ elementar yoyning uzunligini belgilab, bu yoy massasi uchun

$$m_i \approx \mu(M_i)\Delta s_i, \;\; i = 1,2, \dots, n$$

taqribiy tenglikni yozish mumkin. U holda AB egri chiziqning m massasi uchun

$$m = \sum_{i=1}^{n} \mu(M_i)\Delta s_i$$

taqribiy tenglik o'rinli bo'ladi. Elementar yoylarning Δs_i uzunligi qanchalik kichik bo'lsa bu taqribiy tenglikdagi xatolik shunchalik kichik bo'ladi. Shuning uchun AB egri chiziqning m massasi sifatida

$$m = \lim_{\lambda \to 0} \sum_{i=1}^{n} \mu(M_i)\Delta s_i$$

limitning qiymatini olish mumkin.

Egri chiziq massasining mazkur ta'rifini I tur egri chiziqli integral ta'rifi bilan taqqoslab

$$m = \int_{AB} \mu(M)ds = \int_{AB} \mu(x,y)ds \qquad (9.14)$$

formulani hosil qilamiz. Fazoviy AB egri chiziq bo'lganda uning massasi uchun shunga o'shash

$$m = \int\limits_{AB} \mu(M)ds = \int\limits_{AB} \mu(x,y,z)ds \tag{9.15}$$

formula o‘rinli bo‘ladi.

9.5-Misol. Vint chizig‘ining P nuqtadagi $\mu(P)$ zichligi bu nuqtaning radius-vektoriga praporsional bo‘lsa, chiziq bitta buramasining m massasini toping.

► Biz yuqorida bu chiziqni qaradik (9.4-rasm). Uning bitta buramasi $x = a\cos t$, $y = a\sin t$, $z = ht$ tengliklar va $0 \le t \le 2\pi$ shart bilan $P(x,y,z)$ nuqtadagi zichlik esa shartga ko‘ra $\mu(P) = kr = k\sqrt{x^2+y^2+z^2}$ tenglik bilan aniqlanadi, bu yerda k −praporsionallik koeffisiyenti. U holda (9.13) va (9.15) formulalarga ko‘ra

$$m = \int\limits_L \mu(x,y,z)ds = k\int\limits_L \sqrt{x^2+y^2+z^2}ds =$$

$$= \int\limits_0^{2\pi} k\sqrt{a^2+h^2}\sqrt{a^2+h^2t^2}dt =$$

$$= k\sqrt{a^2+h^2}\left(\frac{t}{2}\sqrt{a^2+h^2t^2} + \frac{a^2}{2h}\ln\left(ht+\sqrt{a^2+h^2t^2}\right)\right)\Bigg|_0^{2\pi} =$$

$$= k\sqrt{a^2+h^2}\left(\pi\sqrt{a^2+4\pi^2h^2} + \frac{a^2}{2h}\ln\frac{2\pi h+\sqrt{a^2+4\pi^2h^2}}{a}\right). \quad ◄$$

Moddiy chiziqning statik va inersiya momentlari va og‘irlik markazining koordinatalari. Fazoviy AB egri ciziqning zichligi $\mu(x,y,z)$ funksiya bilan berilgan bo‘lsin. AB chiziqni $A_0 = A$, A_1,…, $A_n = B$ nuqtalar yordamida uzunliklari Δs_i bo‘lgan n ta $A_{i-1}A_i$ elementar yoylarga bo‘lamiz. Bo‘linishlar shunchalik kichkki, bunda bitta elementar yoydagi zichlik o‘zgarmas va u bu elementar yoyning $M_i(x_i,y_i,z_i)$ nuqtasidagi $\mu(x_i,y_i,z_i)$ zichlikka teng deb faraz qilamiz. Bu holda $A_{i-1}A_i$ elementar yoyni uning $M_i(x_i,y_i,z_i)$ nuqtasi bilan almashtirish mumkin va bu nuqtada butun $\mu(x_i,y_i,z_i)\Delta s_i$ massa jamlangan bo‘ladi. Shuning uchun AB egri chiziqning Oxy, Oyz, Oxz tekislikliklarga nisbatan statik momentlarini

$$S_{Oxy} \approx \sum_{i=1}^{n} z_i\mu(x_i,y_i,z_i)\Delta s_i,\ S_{Oyz} \approx \sum_{i=1}^{n} x_i\mu(x_i,y_i,z_i)\Delta s_i,\ S_{Oxz} \approx \sum_{i=1}^{n} y_i\mu(x_i,y_i,z_i)\Delta s_i$$

taqribiy tengliklar bilan aniqlash mumkin. Bu yerda ham $\lambda = \max_{1\le i\le n}\{\Delta s_i\}$ eng katta yoy uzunligini nolga intiltirib statik va inersiya momentlarini I tur egri chiziqli integral yordamida hisoblanuvchi

$$S_{Oxy} = \int_{AB} z\mu(x,y,z)ds, \quad S_{Oyz} = \int_{AB} x\mu(x,y,z)ds,$$

$$S_{Oxz} = \int_{AB} y\mu(x,y,z)ds \qquad (9.16)$$

formulalarni hosil qilamiz. Xuddi shu singari mulohaza yuritib, AB egri chiziqning bu tekisliklarga nisbatan inersiya momentlar uchun mos ravishda

$$I_{Oxy} = \int_{AB} z^2\mu(x,y,z)ds, I_{Oyz} = \int_{AB} x^2\mu(x,y,z)ds,$$

$$I_{Oxz} = \int_{AB} y^2\mu(x,y,z)ds \qquad (9.17)$$

formulalarni hosil qilish mumkin.

AB egri chiziqning Ox o‘qqa nisbatan I_x inersiya momenti $I_x = I_{Oxy} + I_{Oxz}$ tenglik vilan hisoblanadi, boshqacha qilib aytganda

$$I_x = \int_{AB} (z^2 + y^2)\mu(x,y,z)ds \qquad (9.18)$$

formula bilan hisoblanadi. Xuddi shu singari Oy va Oz o‘qlarga nisbatan inersiya momentlari uchun

$$I_y = \int_{AB} (x^2 + z^2)\mu(x,y,z)ds, I_z = \int_{AB} (x^2 + y^2)\mu(x,y,z)ds \qquad (9.19)$$

formulalar o‘rinli.

AB egri chiziq $C(x_c, y_c, z_c)$ og‘irlik markazining koordinatalari

$$x_c = \frac{S_{Oyz}}{m}, y_c = \frac{S_{Oxz}}{m}, z_c = \frac{S_{Oxy}}{m}$$

tengliklarga ko‘ra topiladi. (9.15) va (9.16) formulalarni inobatga olsak

$$x_c = \frac{\int_{AB} x\mu(x,y,z)ds}{\int_{AB} \mu(x,y,z)ds}, y_c = \frac{\int_{AB} y\mu(x,y,z)ds}{\int_{AB} \mu(x,y,z)ds},$$

$$z_c = \frac{\int_{AB} z\mu(x,y,z)ds}{\int_{AB} \mu(x,y,z)ds} \qquad (9.20)$$

formulalarga ega bo‘lamiz.

$\mu(x,y)$ zichlik funksiyaga ega bo‘lgan Oxy tekislikda yotuvchi AB egri chiziq uchun Ox va Oy o‘qlarga nisbatan statik va inersiya momentlari mos ravishda

$$S_x = \int_{AB} y\mu(x,y)ds,\ I_x = \int_{AB} y^2\mu(x,y)ds,$$
$$S_y = \int_{AB} x\mu(x,y)ds,\ I_y = \int_{AB} x^2\mu(x,y)ds$$

formulalarga ko‘ra hisoblanadi. Uning og‘irlik markazi koordinatalari esa

$$x_c = \frac{\int_{AB} x\mu(x,y)ds}{\int_{AB} \mu(x,y)ds},\ y_c = \frac{\int_{AB} y\mu(x,y)ds}{\int_{AB} \mu(x,y)ds}$$

formulalar bilan topiladi.

9.6-Misol. $x = \cos t$, $y = \sin t$, $z = t$ tengliklar bilan aniqlanuvchi vint chizig‘ining $P(x,y,z)$ nuqtadagi zichligi $\mu(x,y,z) = kz$ funksiya bilan aniqlansa, chiziq birinchi buramasining og‘irlik markazi koordinatalarini toping.

► Hosilalarni topamiz: $x' = -\sin t$, $y' = \cos t$, $z' = 1$. U holda (9.15) formulaga ko‘ra cgiziq massasini hisoblaymiz

$$m = \int_{AB} kzds = \int_0^{2\pi} kt\sqrt{\sin^2 t + cos^2 t + 1}dt =$$
$$= \sqrt{2}k\int_0^{2\pi} tdt = \sqrt{2}k\left.\frac{t^2}{2}\right|_0^{2\pi} = 2\sqrt{2}k\pi^2.$$

(9.16) formulalarga ko‘ra chiziqning Oxy, Oyz va Oxz tekisliklarga nisbatan statik momentlarni hisoblaymiz

$$S_{Oxy} = \int_{AB} z\mu(x,y,z)ds = \int_{AB} kz^2ds = \int_0^{2\pi} kt^2\sqrt{\sin^2 t + cos^2 t + 1}dt =$$
$$= \sqrt{2}k\int_0^{2\pi} t^2dt = \sqrt{2}k\left.\frac{t^3}{3}\right|_0^{2\pi} = \frac{8\sqrt{2}k\pi^3}{3};$$

$$S_{Oyz} = \int_{AB} x\mu(x,y,z)ds = \int_{AB} xkzds = \int_0^{2\pi} kt\cos t\sqrt{2}dt =$$
$$= k\sqrt{2}\int_0^{2\pi} t\cos t\,dt = \begin{vmatrix} u = t, & du = dt \\ dv = \cos t\,dt, & v = \sin t \end{vmatrix} = k\sqrt{2}t\sin t|_0^{2\pi} -$$
$$-k\sqrt{2}\int_0^{2\pi} \sin t\,dt = k\sqrt{2}\cos t|_0^{2\pi} = 0;$$

$$S_{Oxz} = \int_{AB} y\mu(x,y,z)ds = \int_{AB} ykzds = \int_0^{2\pi} kt\sin t\sqrt{2}dt =$$

$$= k\sqrt{2}\int_0^{2\pi} t\sin t\,dt = \begin{vmatrix} u = t, & du = dt \\ dv = \sin t\,dt, & v = -\cos t \end{vmatrix} =$$

$$= -k\sqrt{2}t\cos t\big|_0^{2\pi} + k\sqrt{2}\int_0^{2\pi} \cos t\,dt = -2\sqrt{2}k\pi.$$

Og‘irlik markazining koordinatalarini (9.20) formulaga ko‘ra topamiz:

$$x_c = \frac{S_{Oyz}}{m} = \frac{0}{2\sqrt{2}k\pi^2} = 0, y_c = \frac{S_{Oxz}}{m} = \frac{-2\sqrt{2}k\pi}{2\sqrt{2}k\pi^2} = -\frac{1}{\pi},$$

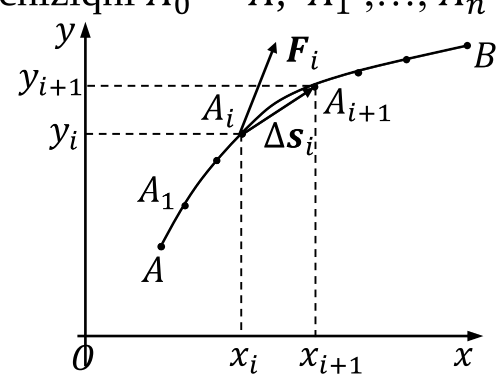
$$z_c = \frac{S_{Oxy}}{m} = \frac{\frac{8\sqrt{2}k\pi^3}{3}}{2\sqrt{2}k\pi^2} = \frac{4}{3}\pi$$

Shunday qilib, vint chizig‘i birinchi buramasining og‘irlik markazi $C(0, -1/\pi, 4\pi/3)$ nuqtada ekan.◀

9.6. II tur egri chiziqli integral

Kuchning bajargan ishi. Moddiy nuqta biror egri chiziq bo‘ylab harakatlanganda unga ta’sir qiluvchi o‘zgaruvchan kuchning bajargan ishini hisoblash II tur egri chiziqli integral tushunchasiga olib keladi.

$M(x, y)$ nuqta AB egri chiziq boylab A nuqtadan B nuqtaga qarab $\boldsymbol{F}$ kuch ta’sirida harakatlanayotgan bo‘lsin. $\boldsymbol{F}$ kuch M nuqta harakatlanganda qiymati bo‘yicha ham, yo‘nalishi bo‘yicha ham o‘zgaradi, ya’ni M nuqtaning funksiyasidan iborat: $\boldsymbol{F} = \boldsymbol{F}(M)$. M nuqta A holatdan B holatga o‘tganda $\boldsymbol{F}$ kuch bajargan $\tilde{A}$ ishni hisoblaymiz (9.6-rasm). Buning uchun AB egri chiziqni $A_0 = A,\ A_1 ,\dots, A_n = B$



9.6-rasm

nuqtalar yordamida A nuqtadan B nuqtaga qarab n ta qismga ajratamiz va $\overrightarrow{A_iA_{i+1}}$ vektorni $\Delta\boldsymbol{s}_i$ orqali belgilaymiz. $\boldsymbol{F}$ kuchning A_i nuqtadagi qiymatini $\boldsymbol{F}_i$ orqali belgilaymiz. U holda $\boldsymbol{F}_i\Delta\boldsymbol{s}_i$ skalyar ko‘paytmani $\boldsymbol{F}$ kuchning $\overset{\frown}{A_iA_{i+1}}$ yoy boylab bajargan ishining taqribiy qiymati sifatida olish mumkin:

$$\tilde{A}_i \approx \boldsymbol{F}_i\Delta\boldsymbol{s}_i.$$

$P(x, y)$ va $Q(x, y)$ funksiyalar $\boldsymbol{F}$ kuchning mos ravishda Ox va Oy o‘qlardagi proyeksiyalari bo‘lsin, ya’ni

$$\boldsymbol{F} = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}.$$

Δx_i va Δy_i orqali x_i va y_i koordinatalarning A_i nuqtadan A_{i+1} nuqtaga o‘tganda olgan orttirmalarini belgilaymiz, ya’ni $\Delta x_i = x_{i+1} - x_i$, $\Delta y_i = y_{i+1} - y_i$ va natijada

$$\Delta \boldsymbol{s}_i = \Delta x_i \boldsymbol{i} + \Delta y_i \boldsymbol{j}$$

tenglikka ega bo‘lamiz. U holda

$$\boldsymbol{F}_i \Delta \boldsymbol{s}_i = P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i$$

bo‘ladi. Shuning uchun $\boldsymbol{F}$ kuchning butun AB egri chiziq bo‘ylab bajargan $\tilde{A}$ ishi taqriban

$$\tilde{A} \approx \sum_{i=0}^{n-1} \boldsymbol{F}_i \Delta \boldsymbol{s}_i = \sum_{i=0}^{n-1} [P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i] \qquad (9.21)$$

yig‘indiga teng bo‘ladi.

II tur egri chiziqli integral. Oxy tekislikda AB egri chiziq berilgan va $P(x,y)$, $Q(x,y)$ funksiyalar bu egri chiziqda aniqlangan bo‘lsin. AB egri chiziqni $A_0 = A$, A_1, …, $A_n = B$ nuqtalar yordamida A nuqtadan B nuqtaga qarab yo‘nalishda uzunliklari Δs_i $(i = 0,2, \dots, n-1)$ bo‘lgan n ta $\overset{\frown}{A_\iota A}_{i+1}$ yoylarga ajratamiz. Har bir $\overset{\frown}{A_\iota A}_{i+1}$ elementar yoyda ixtiyoriy ravishda $M_i(x_i^*, y_i^*)$ nuqta tanlaymiz va

$$\sum_{i=0}^{n-1} P(x_i^*, y_i^*)\Delta x_i, \quad \sum_{i=0}^{n-1} Q(x_i^*, y_i^*)\Delta x_i \qquad (9.22)$$

yig‘indilarni tuzamiz, bu yerda $\Delta x_i = x_{i+1} - x_i$, $\Delta y_i = y_{i+1} - y_i$ bo‘lib, ular $\overset{\frown}{A_\iota A}_{i+1}$ yoyning mos ravishda Ox va Oy o‘qlardagi proyeksiyalari (9.6-rasm).

(9.22) yig‘indilar mos ravishda $P(x,y)$ funksiyaning x o‘zgaruvchi bo‘yicha va $Q(x,y)$ funksiyaning y o‘zgaruvchi bo‘yicha integral yig‘indilari deb ataladi.

9.2-Ta’rif. Agar Δs_i yoy uzunliklarining $\lambda = \max\limits_{0 \le i \le n-1} \Delta s_i$ eng kattasi nolga intilganda (9.22) yig‘indilar chekli limitga intilsa va bu limit AB yoyning bo‘linish usuliga va $M_i(x_i^*, y_i^*)$ nuqtalarning tanlanishiga bog‘liq bo‘lmasa, ularni mos ravishda $P(x,y)$ funksiyadan x o‘zgaruvchi bo‘yicha va $Q(x,y)$ funksiyadan y o‘zgaruvchi bo‘yicha olingan *II tur egri chiziqli integrallar* deb ataymiz va ullarni mos ravishda

$$\int_{AB} P(x,y)dx \text{ va } \int_{AB} Q(x,y)dy$$

orqali belgilaymiz.

Shunday qilib

$$\int_{AB} P(x,y)dx = \lim_{\lambda\to 0}\sum_{i=0}^{n-1} P(x_i^*, y_i^*)\Delta x_i$$

va

$$\int_{AB} Q(x,y)dy = \lim_{\lambda\to 0}\sum_{i=0}^{n-1} Q(x_i^*, y_i^*)\Delta y_i.$$

Umumiy ko‘rinishdagi $\int_{AB} P(x,y)dx + Q(x,y)dy$ II tur egri chiziqli integral

$$\int_{AB} P(x,y)dx + Q(x,y)dy = \int_{AB} P(x,y)dx + \int_{AB} Q(x,y)dy$$

tenglik bilan aniqlanadi.

Endi kuchning bajargan ishi masalasiga qaytadigan bo‘lsak (9.21) taqribiy tenglikning o‘ng tomonida $P(x,y)$ va $Q(x,y)$ funksiyalarning AB egri chiziq bo‘ylab umumiy ko‘rinishdagi II tur egri chiziqli integral yig‘indilari turibdi. Shuning uchun yoy uzunliklarining $\lambda = \max_{0\le i\le n-1} \Delta s_i$ eng kattasi nolga intilganda bu taqribiy tenglik aniq tenglikka aylanadi va $\boldsymbol{F} = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ kuch bajargan ishning qiymati

$$\tilde{A} = \int_{AB} P(x,y)dx + Q(x,y)dy$$

tenglik bilan topiladi.

Fazoviy AB egri chiziq boylab $\int_{AB} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ egri chiziqli integral xuddi shu singari aniqlanadi.

II tur egri chiziqli integralning mavjudligi va uni hisoblash. AB yassi chiziq

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} t \in [\alpha, \beta]$$

parametrik tenglama bilan berilgan bo‘lsin, bu yerda $x(t), y(t)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz. Parametrning $t=\alpha$ qiymatiga A nuqta, $t=\beta$ qiymatiga esa B nuqta mos keladi deb faraz qilamiz.

9.2-Teorema. Agar AB egri chiziqni tasvirlovchi $x(t)$ va $y(t)$ funksiyalar uzluksiz differensiallanuvchi, $P(x,y)$ va $Q(x,y)$ funksiyalar esa AB egri chizqda uzluksiz bo‘lsa, $P(x,y)$ va $Q(x,y)$ funksiyalardan AB egri chizq boylab olingan umumiy ko‘rinishdagi II tur egri chiziqli integral mavjud va uning uchun

$$\int_{AB} P(x,y)dx + Q(x,y)dy =$$

$$=\int_{\alpha}^{\beta}\left[P\big(x(t),y(t)\big)x'(t)+Q\big(x(t),y(t)\big)y'(t)\right]dt \qquad (9.23)$$

tenglik o‘rinli bo‘ladi.

►Umumiy ko‘rinishdagi II tur egri chiziqli integral $P(x,y)$ va $Q(x,y)$ funksiyalar aniqlaydigan ikki qismdan iborat. Teoremaning isbotini bitta qism bilan chegaralanish mumkin, ikkinchi qismi uchun isbot xuddi shu singari bo‘ladi.

Shunday qilib, silliq AB egri chiziqda uzluksiz bo‘lgan $P(x,y)$ funksiya uchun II tur egri chiziqli integral mavjud va uning uchun

$$\int_{AB}P(x,y)dx=\int_{\alpha}^{\beta}P\big(x(t),y(t)\big)x'(t)dt \qquad (9.24)$$

tenglik o‘rinli bo‘lishini isbotlaymiz.

AB egri chiziqni ixtiyoriy $A_0=A$, A_1, …, $A_n=B$ nuqtalar yordamida $\overset{\frown}{A_iA}_{i+1}$ elementar yoylarga ajratuvchi bo‘linishni tanlaymiz. A_i bo‘linish nuqtalariga t parametrning $t_i, i=0,1,\dots,n$ qiymatlari mos kelsin. Bu qiymatlar uchun

$$t_0=\alpha<t_1<t_2<\cdots<t_{n-1}<t_n=\beta$$

munosabatlar o‘rinli bo‘ladi. Har bir $\overset{\frown}{A_iA}_{i+1}$ elementar yoyda ixtiyoriy ravishda $M_i(x_i^*,y_i^*)$ nuqta tanlaymiz va τ_i −parametrning bu nuqtaga mos keluvchi qiymati bo‘lsin. $t_i\le\tau_i\le t_{i+1}$ ekanligi ravshan. Tanlangan bo‘linishga mos keluvchi S integral yig‘indini

$$S=\sum_{i=0}^{n-1}P(x_i^*,y_i^*)\Delta x_i=\sum_{i=0}^{n-1}P(x(\tau_i),y(\tau_i))(x(t_{i+1})-x(t_i))=$$

$$=\sum_{i=0}^{n-1}P(x(\tau_i),y(\tau_i))\int_{t_i}^{t_{i+1}}x'(t)dt=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}P(x(\tau_i),y(\tau_i))x'(t)dt$$

ko‘rinshda yozib olish mumkin.

(9.24) tenglikning o‘ng tomonidagi I integral teorema shartiga ko‘ra mvjud, chunki integral ostidagi funkisya $[\alpha,\beta]$ kesmada uzluksiz. Aniq integralning additivlik xossasidan foydalanib, bu integralni

$$I=\int_{\alpha}^{\beta}P\big(x(t),y(t)\big)x'(t)dt=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}P\big(x(t),y(t)\big)x'(t)dt$$

ko‘rinishda yozamiz. S integral yig‘indi va I integral uchun hosil qilingan tasvirlardan

$$S-I=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\Big(P\big(x(\tau_i),y(\tau_i)\big)-P\big(x(t),y(t)\big)\Big)x'(t)dt$$

tenglikka ega bo‘lamiz. Bu tenglikning o‘ng tomonini baholaymiz.

$F(t)=P\big(x(t),y(t)\big)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz, shuning uchun u bu kesmada tekis uzluksiz hamdir. Demak, ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $|t''-t'|<\delta$ tengsizlikni qanoatlantiruvchi barcha $t',t''\in[\alpha,\beta]$ qiymatlar uchun $|F(t')-F(t'')|<\varepsilon$ tengsizlik o‘rinli bo‘ladi. A boshlang‘ich nuqtadan AB egri chiziq yoyi o‘zgaruvchan uzunlikni hisoblovchi $s(t)$ funksiya uzluksiz differensiallanuvchi va o‘suvchi bo‘ladi. Demak uning $[0,s_{AB}]$ kesmada aniqlangan $t(s)$ teskari funksiyasi mavjud, bu yerda $s_{AB}-$qiymat AB egri chiziq uzunligi. Bu funksiya uzluksiz differensiallanuvchi, uning $|t'(s)|$ hosilasining moduli $[0,s_{AB}]$ kesmada K_1 maksimumga erishadi, ya’ni $|t'(s)|\le K_1, 0\le s\le s_{AB}$. U holda (5.65) chekli ayirmalar formulasiga ko‘ra

$$|t_{i+1}-t_i|=|t'(\xi_i)|\Delta s_i\le K_1\Delta s_i\le K_1\lambda,$$

bu yerda $s_i=s(t_i), i=0,1,2,\dots,n$, $\Delta s_i=s_{i+1}-s_i-$ elementar yoy uzunliklari va $\lambda=\max\limits_{0\le i\le n-1}\Delta s_i$.

Tanlangan bo‘linish $\lambda<\delta(\varepsilon)/K_1$ shartni qanoatlantirsin. U holda ixtiyoriy $t',t''\in[t_i,t_{i+1}]$ qiymatlar uchun

$$|t'-t''|\le t_{i+1}-t_i\le K_1\Delta s_i\le\delta(\varepsilon).$$

Demak, $t\in[t_i,t_{i+1}]$ bo‘lsa $|F(\tau_i)-F(t)|<\varepsilon$ bo‘ldi va

$$|S-I|\le\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\Big|P\big(x(\tau_i),y(\tau_i)\big)-P\big(x(t),y(t)\big)\Big||x'(t)|dt<$$

$$<\varepsilon K_2\sum_{i=0}^{n-1}\Delta t_i=\varepsilon K_2(\beta-\alpha),$$

bu yerda K_2- uzluksiz $|x'(t)|$ funksiyaning $[\alpha,\beta]$ kesmadagi maksimal qiymati. Shunday qilib $\lim\limits_{\lambda\to 0}S=I$ ekan, bu esa teoremaning to‘g‘riligini tasdiqlaydi.◄

Agar AB egri chiziq $y=\varphi(x), x\in[a,b]$ tenglama bilan berilgan va $\varphi(x)$ funksiya va uning $\varphi'(x)$ hosilasi $[a,b]$ kesmada uzluksiz bo‘lsa, x o‘zgaruvchini parametr sifatida olamiz, natijada AB egri chiziqning parametrik tenglamasi: $x=x, y=\varphi(x), x\in[a,b]$ ko‘rinishda bo‘ladi. U holda (9.23) formuladan

$$\int_{AB} P(x,y)dx + Q(x,y)dy = \int_a^b \left[P\big(x,\varphi(x)\big) + Q\big(x,\varphi(x)\big)\varphi'(x)\right]dx \tag{9.25}$$

formulani hosil qilamiz. Xususiy holda

$$\int_{AB} P(x,y)dx = \int_a^b P\big(x,\varphi(x)\big)dx \tag{9.26}$$

bo‘ladi.

Agar fazoviy silliq AB egri chiziq $x = x(t), y = y(t), z = z(t), t \in [\alpha, \beta]$ tenglamalar bilan berilgan bo‘lsa $\int_{AB} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ egri chiziqli integral

$$\int_{AB} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int_\alpha^\beta [P\big(x(t),y(t),z(t)\big)x'(t) + Q\big(x(t),y(t),z(t)\big)y'(t) +$$

$$+R(x(t),y(t),z(t))z'(t)]dt \tag{9.27}$$

formula bilan hisoblanadi.

9.7-Misol. $\int_{AB} yzdx + xzdy + xydz$ egri chiziqli integralni hisoblang, bu yerda integrallash $x = t$, $y = t^2$, $z = t^3$ egri chiziqning $A(1,1,1,)$ va $B(2,4,8)$ nuqtalar orasidagi qismi boyicha olingan.

►Hosilalarni hisoblaymiz: $x'(t) = 1$, $y'(t) = 2t$, $z'(t) = 3t^2$. U holda (9.27) formulaga ko‘ra

$$\int_{AB} yzdx + xzdy + xydz = \int_1^2 (t^2 \cdot t^3 \cdot 1 + t \cdot t^3 \cdot 2t + t \cdot t^2 \cdot 3t^2)dt =$$

$$= \int_1^2 (t^5 + 2t^5 + 3t^5)dt = \int_1^2 6t^5 dt = t^6|_1^2 = 63.◄$$

9.8-Misol. $\int_{AB} (x-y)dx + dy + zdz$ egri chiziqli integralni hisoblang, bu yerda AB chiziq $z = x^2 + y^2$ parabolid bilan $z = 4$ tekislik kesishuvidan hosil bo‘lgan chiziqning $A(2,0,4)$ va $B(-2,0,4), (y \geq 0)$ nuqtalar orasidagi qismi.

►Aytib o‘tilgan kesimda

$$\begin{cases} x^2 + y^2 = 4, \\ z = 4 \end{cases}$$

aylana hosil bo‘ladi. Shuning uchun aylananing parametrik tenglamasi

$$\begin{cases} x = 2\cos t\,, \\ y = 2\sin t\,, \\ z = 4 \end{cases}$$
ko‘rinishda bo‘ladi. Endi parametrning α va β integrallash chegaralarini
$$\begin{cases} 2 = 2\cos\alpha\,, \\ 0 = 2\sin\alpha\,, \\ 4 = 4 \end{cases} \quad \begin{cases} -2 = 2\cos\beta\,, \\ 0 = 2\sin\beta\,, \\ 4 = 4 \end{cases}$$
shartlardan topamiz. $y \geq 0$ ekanligini inobatga olsak $\alpha = 0$ va $\beta = \pi$ qiymqtlarga ega bo‘lamiz.

Parametrik funksiyalarning hosilalri esa $x'(t) = -2\sin t\,, y'(t) = 2\cos t\,,\ z'(t) = 0$ bo‘ladi. U holda (9.27) formulaga ko‘ra
$$\int_{AB}(x-y)dx + dy + zdz = \int_0^{\pi}\big((2\cos t - 2\sin t)(-2\sin t) + 2\cos t\big)dt =$$
$$= 2\int_0^{\pi}(-2\cos t\sin t + 2\sin^2 t + \cos t)dt =$$
$$= 2\left(\cos^2 t + t - \frac{1}{2}\sin 2t + \sin t\right)\Big|_0^{\pi} = 2\pi. ◄$$

9.9-Misol. $\int_{AB} xydx + x^2dy$ egri chiziqli integralni $A(1,1)$ va $B(3,9)$ nuqtalarni tutashtiruvchi $y = x^2$ parabola bo‘yicha hisoblang.

►Parabolani ifodalovchi funksiyaning hosilasi $y'(x) = 2x$ bo‘ladi. U holda (9.25) formulaga ko‘ra
$$\int_{AB} xydx + x^2dy = \int_1^3(x\cdot x^2 + x^2\cdot 2x)dx = \int_1^3 3x^3dx = \frac{3}{4}x^4\Big|_1^3 =$$
$$= \frac{3}{4}(81-1) = 60. ◄$$

II tur egri chiziqli integralning xossalari.

1. AB egri chiziqda yo‘nalish o‘zgarganda II tur egri chiziqli integral o‘z ishorasini o‘zgartiradi:
$$\int_{AB} P(x,y)dx + Q(x,y)dy = -\int_{BA} P(x,y)dx + Q(x,y)dy.$$

2. O‘zgarmas ko‘paytuvchini egri chiziqli integral ostidan chiqarish mumkin:
$$\int_{AB} k\cdot P(x,y)dx + k\cdot Q(x,y)dy = k\int_{AB} P(x,y)dx + Q(x,y)dy.$$

3. Ikkita funksiya yig‘indisining egri chiziqli integrali bu funksiyalar egri chiziqli integrallarining yig‘indisiga teng:

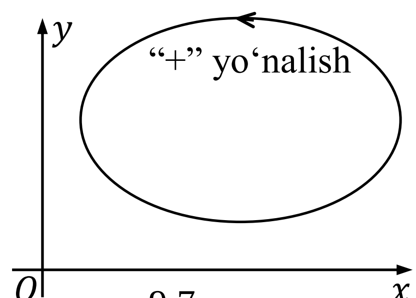
$$\int_{AB} (P_1(x,y) + P_2(x,y))dx = \int_{AB} P_1(x,y)dx + \int_{AB} P_2(x,y)dx,$$
$$\int_{AB} (Q_1(x,y) + Q_2(x,y))dy = \int_{AB} Q_1(x,y)dy + \int_{AB} Q_2(x,y)dy.$$

2 va 3xossalar II tur egri chiziqli integralning chiziqliligini anglatad: funksiyalar chiziqli kombinatsiyasining integrali har bir funksiya integrallarining xuddi shunday chiziqli kombinatsiyasiga teng.

4. Agar AB egri chiziq ketma-ket keladigan chekli sondagi yoylarga ajratilgan bo‘lsa va ana shu yoylarning har birida II tur egri chiziqli integral mavjud bo‘lsa, butun AB egri chiziqda ham integral mavjud va u alohida qismiy yoylar bo‘yicha olingan integrallar yig‘indisiga teng.

1-4 xossalarni integral yig‘indilarning limiti sifatida II tur egri chiziqli integralning ta’rifidan va limitlarning ma’lum xossalaridan foydalanib isbotlash unchalik qiyin emas. Bu isbotlar aniq integralning mos xossalarining isbotini takrorlaydi. Shuning uchun biz ularda to‘xtalib o‘tirmaymiz.

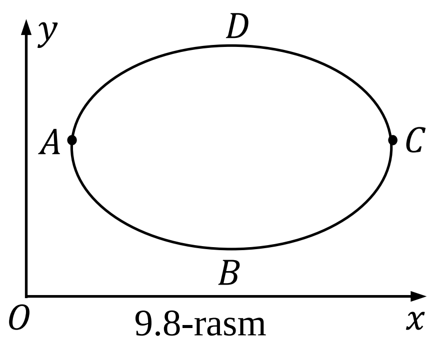
Yopiq L kontur bo‘yicha II tur egri chiziqli integral maxsus $\oint$ simvol bilan belgilanadi. Bunday integrallarda boshlang‘ich va oxirgi nuqtalarni ko‘rsatib integrallash yo‘nalishini aniqlab bo‘lmaydi. Konturni aylanib chiqish yo‘nalishini aniqlashni turli usullar bilan berish mumkin. Masalan, kontur parametrik tenglamalar bilan berilganda parametrning o‘sishiga mos keluvchi yo‘nalishni tanlash mumkin.



9.7-rasm

Sodda yassi konturlarda (aylana, ellips) aylanish yo‘nalishini soat strelkasi yo‘nalishi bilan taqqoslash mumkin. Bunda soat strelkasi harakatiga qarama-qarshi yo‘nalishni musbat, coat strelkasi harakati yo‘nalishini esa manfiy yo‘nalish deb ataladi (9.7-rasm).

Musbat yo‘nalishli konturdagi integrallar uchun ba’zan maxsus $\ointctrclockwise$ simvoldan, manfiy yo‘nalishli konturlar uchun esa $\ointclockwise$ simvoldan foydalaniladi.



9.8-rasm

5. Yopiq kontur bo‘yicha olingan egri chiziqli integral boshlang‘ich nuqtaning tanlanishiga bog‘liq emas (faqat konturni aylanib chiqish yo‘nalishigagina bog‘liq).

►Haqiqattan ham,

$$\oint\limits_{ABCDA} = \oint\limits_{ABC} + \oint\limits_{CDA}$$

(9.8-rasm). Boshqa tomondan

$$\oint\limits_{CDABC} = \oint\limits_{CDA} + \oint\limits_{ABC}.$$

Shunday qilib,

$$\oint\limits_{ABCDA} = \oint\limits_{CDABC}. \blacktriangleleft$$

9.7. Ostrogradskiy-Grin formulasi

D soha bo‘yicha ikki o‘lchovli integral bilan bu sohaning L chegarasi bo‘yicha II tur egri chiziqli integral orasidagi bog‘liqlikni ifodalovchi Ostrogradskiy-Grin formulasi matematk tahlilda juda ko‘p tadbiqqa ega.

Oxy tekislikda to‘g‘ri D soha L egri chiziq bilan chegaralangan bo‘lsin.

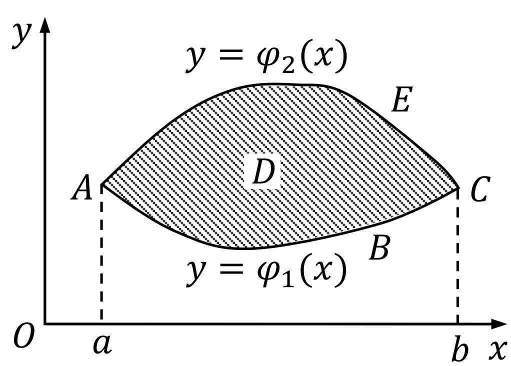
9.3-Teorema. Agar $P(x,y)$ va $Q(x,y)$ funksiyalar o‘zlarining $\frac{\partial P}{\partial y}$ va $\frac{\partial Q}{\partial x}$ xususiy hosilalari bilan birga D sohada uzluksiz bo‘lsa,

$$\iint\limits_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint\limits_{L} Pdx + Qdy \qquad (9.28)$$

tenglik o‘rinli bo‘ladi, bu yerda L −integrallash chizig‘i D sohaning chegarasi va integrallash musbat yo‘nalishda (ya’ni chiziq bo‘ylab harakatlanganda D soha chap tomonda qoladi) bajariladi.

(9.28) formula Ostrogradskiy-Grin formulasi deb ataladi.

►ABC yoyning tenglamasi $y = \varphi_1(x)$ va ADC yoyning tenglamasi $y = \varphi_2(x)$ bo‘lsin (9.9-rasm). Dastlab $\iint\limits_{D} \frac{\partial P}{\partial y} dxdy$ integralni hisoblaymiz. Ikki o‘lchovli integralni hisoblash qoidasiga ko‘ra



9.9-rasm

$$\iint\limits_D \frac{\partial P}{\partial y}dxdy = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}dy = \int\limits_a^b \left(P(x,y)\Big|_{\varphi_1(x)}^{\varphi_2(x)}\right)dx =$$

$$= \int\limits_a^b P(x,\varphi_2(x))dx - \int\limits_a^b P(x,\varphi_1(x))dx.$$

O‘ng tomondagi ikkala integralga ham (9.26) formulani qo‘llasak

$$\iint\limits_D \frac{\partial P}{\partial y}dxdy = \int\limits_{AEC} P(x,y)dx - \int\limits_{ABC} P(x,y)dx =$$

$$= -\int\limits_{CEA} P(x,y)dx - \int\limits_{ABC} P(x,y)dx = -\oint\limits_L P(x,y)dx \qquad (9.29)$$

Xuddi shu singari

$$\iint\limits_D \frac{\partial Q}{\partial x}dxdy = \oint\limits_L Q(x,y)dy \qquad (9.30)$$

ekanligi isbotlanadi.

Agar (9.30) tenglikdan (9.29) tenglikni hadma-had ayirsak (9.28) tenglik hosil bo‘ladi.◄

Yassi figuraning yuzi. Ostrogradskiy-Grin formulasining bir tadbiqini qaraymiz. $P(x,y) = -y$ va $Q(x,y) = x$ deb olamiz. U holda

$$\frac{\partial Q}{\partial x} = 1, \qquad \frac{\partial P}{\partial y} = -1$$

bo‘ladi va (9.28) Ostrogradskiy-Grin formulasiga ko‘ra

$$\oint\limits_L -ydx + xdy = \iint\limits_D 2dxdy = 2\iint\limits_D dxdy = 2S_D,$$

bu yerda S_D −berilgan D sohaning yuzi.

Bundan esa D yassi sohaning S_D yuzi uchun II tur egri chiziqli integral bilan hisoblanadigan

$$S_D = \frac{1}{2}\oint\limits_L xdy - ydx \qquad (9.31)$$

formulani hosil qilamiz.

9.10-Misol. $L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellips bilan chegaralangan soha yuzini hisoblang.

►Ellipsning tenglamasini parametrik ko‘rinishda yozib olamiz:

$$x = a\cos t, b = b\sin t, 0 \le t \le 2\pi.$$

Qidirilayotgan yuzani (9.31) formulaga ko‘ra hisoblaymiz, bunda egri chiziqli integral konturni musbat yo‘nalishda aylanish bo‘yicha olinadi va bu yo‘nalish t parametrning 0 dan 2π gacha o‘zgarishiga mos keladi. Parametrik funksiyalarning hosilalri

$$x'(t) = -a\sin t,\ y'(t) = b\cos t$$

bo‘lganligi uchun, (9.31) formulaga (9.23) formulani qo‘llaymiz

$$S = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)dt = \pi ab. \blacktriangleleft$$

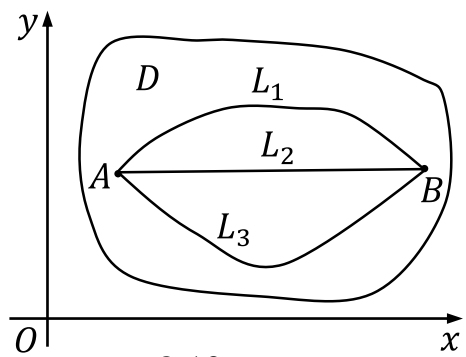
9.8. II tur egri chiziqli integralning integrallash yo‘liga bog‘liq bo‘lmaslik sharti

Oxy tekislikdagi D sohaga tegishli bo‘lgan ixtiyoriy yopiq konturning ichki qismi ham shu D sohaga tegishli bo‘lsa, D soha *bir bog‘lamli soha* deb ataladi.

$A(x_1, y_1)$ va $B(x_2, y_2)$ bir bog‘lamli D sohaning ixtiyoriy nuqtalari bo‘lsin. A va B nuqtalarni turli chiziqlar bilan bog‘lash mumkin (9.10-rasmda L_1, L_2 va L_3). Bu chiziqlarning har biri bo‘yicha

$$I = \int_{AB} P(x,y)dx + Q(x,y)dy$$

integralning qiymati umuman olganda turlicha bo‘ladi.



9.10-rasm

Agar bu qiymat barcha mumkin bo‘lgan AB chiziqlar bo‘yicha bir xil bo‘lsa, I integral *integrallash yo‘liga bog‘liq emas* deb aytiladi. Bu holda I integral uchun uning $A(x_1, y_1)$ boshlang‘ich va $B(x_2, y_2)$ oxirgi nuqtasini ko‘rsatish yetarli va u aniq integralga o‘xshash

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy \qquad (9.32)$$

ko‘rinishda yoziladi.

Qanday shart bajarilganda II tur egri chiziqli integral integrallash yo‘iga bog‘liq bo‘lmaydi?

9.4-Teorema. $P(x,y)$ va $Q(x,y)$ funksiyalar o‘zlarining $\frac{\partial P}{\partial y}$ va $\frac{\partial Q}{\partial x}$ xususiy hosilalari bilan birga bir bog‘lamli D sohada uzluksiz bo‘lsin. U holda quyidagi to‘rtta shart o‘zaro ekvivalent, ya’ni ulardan birining bajarilishi qolganlarining ham bajarilishini keltirib chiqaradi:

(i) D sohada yotgan ixtiyoriy yopiq silliq L kontur uchun

$$\oint_L Pdx + Qdy = 0;$$

(ii)) D sohada yotuvchi ixtiyoriy Ava B nuqtalar uchun

$$\int_{AB} P(x,y)dx + Q(x,y)dy$$

integralning qiymati integrallsh yo‘liga bog‘liq emas;

(iii) $P(x,y)dx + Q(x,y)dy$ ifoda D sohada aniqlangan biror funksiyaning to‘liq differensialidan iborat, ya’ni D sohada aniqlangan shunday $F(x,y)$ funksiya mavjudki, uning uchun

$$dF = P(x,y)dx + Q(x,y)dy$$

tenglik o‘rinli bo‘ladi;

(iv) D sohaning barcha nuqtalarida

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{9.33}$$

tenglik o‘rinli.

►Isbotni (i) → (ii) → (iii) → (iv) →(i) sxema bo‘yicha olib boramiz, ya’ni (i) shatdan (ii) shart, (ii) shartdan (iii) shart, (iii) shartdan (iv) shart va nihoyat (iv) shrtdan (i) shart kelib chiqishini ko‘rsatamiz.

1) (i) → (ii). D sohada A va B nuqtalarni tutashtiruvchi ikkita ixtiyoriy silliq ACB va ADB egri chiziqlarni olamiz (9.11-rasm). Bu ikki chiziqning yig‘indisi D sohada yotuvchi yopiq $L = ACB + BDA$ silliq konturni hosil qiladi. (i) shartga ko‘ra

$$\oint_L Pdx + Qdy = 0.$$

Boshqa tomondan

$$\oint_L Pdx + Qdy = \int_{ACB} P(x,y)dx + Q(x,y)dy +$$
$$+ \int_{BDA} P(x,y)dx + Q(x,y)dy =$$
$$= \int_{ACB} P(x,y)dx + Q(x,y)dy - \int_{ADB} P(x,y)dx + Q(x,y)dy.$$

Nihoyat bundan

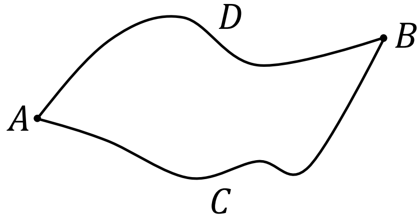
$$\int_{ACB} P(x,y)dx + Q(x,y)dy = \int_{ADB} P(x,y)dx + Q(x,y)dy$$

tenglikka ega bo‘lamiz, ya’ni (ii) shartning bajarilishi kelib chiqadi.

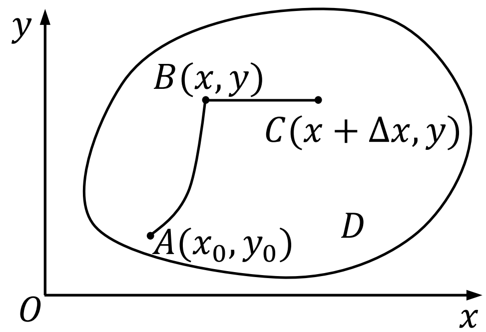
2) (ii) → (iii). $\int_{AB} P(x,y)dx + Q(x,y)dy$ integral integrallash yo‘liga bog‘liq bo‘lmasdan, faqat A va B nuqtalarga bog‘liq bo‘lsin. U holda agar $A =$

$A(x_0, y_0)$ nuqtani o‘zgarmas qilib olsak, bu integral $B(x, y)$ nuqta koordinatalarining funksiyasidan iborat bo‘ladi:

$$\int_{AB} Pdx + Qdy = F(x, y).$$



9.11-rasm



9.12-rasm

Bunday aniqlangan $F(x, y)$ funksiya differensiallanuvchi va uning uchun

$$dF = P(x, y)dx + Q(x, y)dy \tag{9.34}$$

tenglikning bajarilishini ko‘rsatamiz. Buning uchun D sohaning ixtiyoriy B nuqtasida $\frac{\partial F}{\partial x}$ va $\frac{\partial F}{\partial y}$ xususiy hosilalar mavjudligini va

$$\frac{\partial F}{\partial x} = P(x, y), \qquad \frac{\partial F}{\partial y} = Q(x, y) \tag{9.35}$$

tengliklarning bajarilishini ko‘rsatish yetarli.

$F(x, y)$ funksiyaning xususiy hosilalri mavjudligini va (9.35) tengliklarning birinchisi bajarilishini ko‘rsatish uchun $F(x, y)$ funksiyaning x bo‘yicha $B(x, y)$ nuqtadagi xususiy orttirmasini tuzamiz:

$$\Delta_x F = F(x + \Delta x, y) - F(x, y) =$$
$$= \int_{AC} Pdx + Qdy - \int_{AB} Pdx + Qdy = \int_{BC} Pdx + Qdy,$$

bu yerda C nuqta $x + \Delta x$ va y koordinatalarga ega (9.12-rasm). Shartga ko‘ra integral egri chiziq ko‘rinishiga bog‘liq emas, shu sababli integrallash chizig‘i sifatida B va C nuqtalarni tutashtiruvchi kesmani olamiz. U holda

$$\Delta_x F = \int_{BC} Pdx + Qdy = \int_{BC} Pdx = \int_{x}^{x+\Delta x} P(x, y)dx.$$

So‘ngi integralga aniq integraldagi o‘rta qiymat haqidagi teoremani qo‘llaymiz:

$$\Delta_x F = P(x + \theta\Delta x, y)\Delta x, \ 0 < \theta < 1,$$

bundan esa

$$\frac{\Delta_x F}{\Delta x} = P(x + \theta\Delta x, y), \qquad 0 < \theta < 1.$$

Demak, bu yerda limitga o‘tsak

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} P(x + \theta\Delta x, y) = P(x, y).$$

Shartga ko‘ra $P(x,y)$ uzluksiz. $\dfrac{\partial F}{\partial y} = Q(x,y)$ tenglik ham xuddi shu singari isbotalanadi. Shunday qilib, (iii) shartning bajarilishi ko‘rsatildi.

3) (iii) → (iv). D sohada $dF = Pdx + Qdy$ tenglik o‘rinli bo‘ladigan $F(x,y)$ funksiya mavjud bo‘lsin. U holda

$$\frac{\partial F}{\partial x} = P(x,y), \qquad \frac{\partial F}{\partial y} = Q(x,y),$$

va aralsh hosilalar haqidagi 7.6-Teoremaga ko‘ra

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y},$$

ya’ni talab qilingan (9.33) tenglikni hosil qildik.

4) (iv) → (i). Teoremadagi (iv) bajarilgan bo‘lsin va $L -$berilgan D sohada yotuvchi va D^* sohani chegaralovchi bo‘lakli silliq chiziq bo‘lsin. U holda D sohaning bir bog‘lamli ekanligini inobatga olgan holda Grin formulasini qo‘llasak

$$\oint_L Pdx + Qdy = \iint_{D^*} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

tenglikni hosil qilamiz. (iv) shartga ko‘ra o‘ng tomondagi ikki o‘chovli integral nolga teng. Demak D sohada yotuvchi ixtiyoriy L yopiq kontur uchun

$$\oint_L Pdx + Qdy = 0$$

tenglik o‘rinli bo‘ladi degan xulosaga kelamiz.◄

Shunday qilib, $\int_{AB} Pdx + Qdy$ integralning integrallsh yo‘liga bog‘liq bo‘lmaslik sharti sifatida (i), (iii) va (iv) shartlarda ixtiyoriy birini olish mumkin ekan. Lekin ular orasida (9.33) tenglik bilan aniqlanadigan (iv) shart foydalanishga qulay va bajarilishini tekshirish oson. Shuning uchun biz ana shu shartni integrallash yo‘liga bog‘liq bo‘lmaslik sharti deb qabul qilamiz.

Endi (9.32) tenglik bilan aniqlanuvchi integralni hisoblaymiz. (9.33) shart bajarilgan bo‘lsin. U holda 9.4-Teoremaga ko‘ra

$$P(x,y)dx + Q(x,y) = dF(x,y)$$

tenglikni qanotlantiruvchi $F(x,y)$ funksiya mavjud bo‘ladi. Shuning uchun

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy = \int_{(x_1,y_1)}^{(x_2,y_2)} dF(x,y) =$$

$$= F(x,y)\Big|_{(x_1,y_1)}^{(x_2,y_2)} = F(x_2,y_2) - F(x_1,y_1),$$

ya’ni

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy = F(x_2,y_2) - F(x_1,y_1) \tag{9.36}$$

tenglikni hosil qilamiz. (9.36) formula to‘la differensialning egri chiziqli integrali uchun *umumlashgan Nyuton-Lebnis formulasi* deb ataladi.

(9.34) shartni qanoatlantiruvchi $F(x,y)$ funksiyani

$$F(x,y) = \int_{x_0}^{x} P(t,y_0)dt + \int_{y_0}^{y} Q(x,s)ds \tag{9.37}$$

formula yordamida topiladi. Bu yerda (x_0,y_0) boshlang‘ich nuqta sifatida D sohaning ixtiyoriy nuqtasini olish mumkin.

L fazoviy egri chiziq bo‘yicha

$$\int_L Pdx + Qdy + Rdz$$

egri chiziqli integral uchun xuddi shu singari natijalar o‘rinli. Bunda egri chiziqli integralning integrallash yo‘liga bog‘liq bo‘lmaslik sharti

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

ko‘rinishda bo‘ladi. Bu shart bajarilganda $\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} Pdx + Qdy + Rdz$ integral

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} Pdx + Qdy + Rdz = F(x_2,y_2,z_2) - F(x_1,y_1,z_1)$$

formula bilan hisoblanadi. O‘ng tomondagi $F(x,y,z)$ funksiya

$$F(x,y,z) = \int_{x_0}^{x} P(t,y_0,z_0)dt + \int_{y_0}^{y} Q(x,s,z_0)ds + \int_{z_0}^{z} R(x,y,l)dl$$

formulaga ko‘ra aniqlanadi.

9.11-Misol. $I = \int_{(2,2)}^{(3,4)} 2xydx + x^2dy$ integralni hisoblang.

► Bu yerda $P = 2xy, Q = x^2$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$. 9.4-Teoremaga ko‘ra integral integrallash yo‘liga bog‘liq emas. Integrallash yo‘li sifatida $y = 2x - 2$ to‘g‘ri chiziqni olish ham mumkin, yoki (9.36) formuladan foydalanish mumkin. Dastlab boshlang‘ich nuqta sifatida (1,1) nuqtani olib, (9.37) formulaga ko‘ra

$$F(x,y) = \int_1^x 2t \cdot 1dt + \int_1^y x^2 ds = x^2y - 1$$

funksiyani topamiz va nihoyat (9.36) formuladan integralning

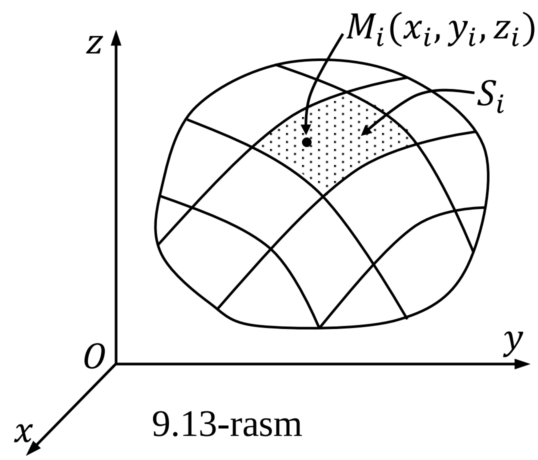
$$I = \int_{(2,2)}^{(3,4)} 2xydx + x^2dy = (x^2y - 1)|_{(2,2)}^{(3,4)} = 28$$

qiymatini topdik.◄

9.9. I tur sirt integrali

Asosiy tushunchalar. Ikki o‘lchovli integralning umulashtirilishi natijasida sirt integrali yuzaga keladi. Bu umumlashtirilish sirt yuzining ta’rifiga asoslanib quriladi.

Agar σ sirtni aniqlovchi $z = z(x,y)$ funksiya uzluksiz xususiy hosilalarga ega bo‘lsa, bu sirtni biz *silliq sirt* deb ataymiz. σ sirt chekli sondagi o‘zaro kesishmaydigan silliq sirtlardan tashkil topgan bo‘lsa, u *bo‘lakli silliq sirt* deb ataladi. Silliq sirtning ixtiyoriy ichki nuqtasida sirtning *urinma va normal tekisliklari* mavjud.



9.13-rasm

$Oxyz$ fazoning yuzasi S ga teng bo‘lgan σ sirti nuqtalarida uzluksiz $f(x,y,z)$ aniqlangan bo‘lsin. σ sirtni yuzalari ΔS_i ga teng bo‘lgan n ta $S_1, S_2, \ldots, S_n$ qismga ajratamiz va ularning diametrlarini d_i $(i = 1,2, \ldots, n)$ orqali belgilaymiz. Har bir S_i qismda ixtiyoriy ravishda bittadan $M_i(x_i, y_i, z_i)$ nuqtani tanlaymiz (9.13-rasm) va

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i \tag{9.38}$$

yig‘indini tuzamiz.

Bu yigindi $f(x,y,z)$ funksiya uchun S sirt boyicha *integral yig‘indi* deb ataladi.

9.3-Ta’rif. Agar $\lambda = \max\limits_{1\le i\le n} d_i$ eng katta diametr nolga intilganda (9.38) integral yig‘indi chekli limitga intilsa va bu limit σ sirtni S_i qismlarga bo‘lish usuliga va bu qismlarda $M_i(x_i,y_i,z_i)$ nuqtalarning tanlanishiga bog‘liq bo‘lmasa, u $f(x,y,z)$ funksiyaning σ sirt bo‘yicha I tur sirt integrali deb ataladi va $\iint\limits_{\sigma} f(x,y,z)ds$ orqali belgilanadi.

Shunday qilib, ta’rifga ko‘ra

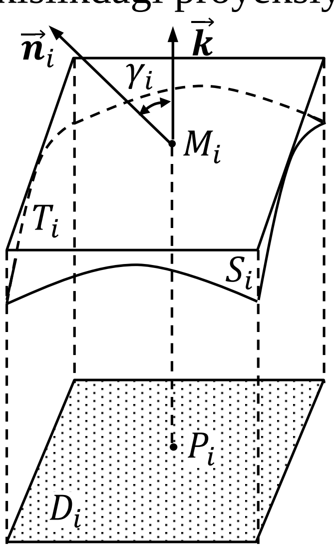
$$\iint\limits_{\sigma} f(x,y,z)ds = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta S_i. \tag{9.39}$$

I tur sirt integralining mavjudligi va uni hisoblash.

9.5-Teorema. Agar $z = z(x,y)$ tenglama bilan berilgan σ sirt silliq, $f(x,y,z)$ funksiya esa bu sirtda uzluksiz bo‘lsa, $f(x,y,z)$ funksiyaning bu sirt bo‘yicha I tur sirt integrali mavjud va

$$\iint\limits_{\sigma} f(x,y,z)ds = \iint\limits_{D} f(x,y,z(x,y))\cdot\sqrt{1+{z'_x}^2+{z'_y}^2}\,dxdy \tag{9.40}$$

tenglik o‘rinli bo‘ladi, bu yerda D integrallash sohasi σ sirtning Oxy tekislikdagi proyeksiyasi.



9.14-rasm

► σ sirtni $S_i, i = 1,2,\dots,n$ qismlarga ajratamiz. D_i orqali S_i qismiy sirtning Oxy tekislikdagi proyeksiyasini belgilaymiz. Natijada σ sirtning Oxy tekislikdagi D proyeksiyasi n ta $D_1, D_2, \dots, D_n$ qismlarga ajraladi. D_i elementar sohada ixtiyoriy $P_i(x_i,y_i)$ nuqtani tanlaymiz va bu nuqtadan Oxy tekislikka perpendikulyar to‘g‘ri chiziq chiqaramiz. Bu to‘g‘ri chiziq σ sirtni S_i qismiy sirtning $M_i(x_i,y_i,z_i)$ nuqtasida kesib o‘tadi (9.14-rasm). Ana shu M_i nuqta orqali σ sirtga urinma tekislik o‘tkazamiz (σ sirt silliq bo‘lganligi uchun bunday urinma tekislik mavjud). Bu tekislikning Oxy tekislikdagi proyeksiyasi D_i elementar soha bo‘ladigan T_i qismini qaraymiz. S_i, T_i va D_i elementar qismlarning yuzalarini mos ravishda ΔS_i, ΔT_i va ΔD_i orqali belgilaymiz. σ sirtni S_i qismlarga yetarlicha kichik qilib bo‘lganda

$$\Delta T_i \approx \Delta S_i \tag{9.41}$$

deb hisoblash mumkin.

γ_i orqali Oz o‘q bilan sirtning M_i nuqtadagi $\vec{\boldsymbol{n}}_i$ normali orasidagi o‘tkir burchakni belgilasak

$$\Delta T_i \cdot \cos\gamma_i = \Delta D_i \tag{9.42}$$

tenglik o‘rinli bo‘ladi (D_i soha T_i sohaning Oxy tekislikdagi proyeksiyasi).

M_i nuqtadagi urinma tekislik tenglamasi ((7.14) formulaga ko‘ra)

$$z'_x(x_i, y_i) \cdot (x - x_i) + z'_y(x_i, y_i) \cdot (y - y_i) - (z - z_i) = 0$$

ko‘rinishda bo‘ladi, shuning uchun $\vec{\boldsymbol{n}}_i$ normal $\left\{z'_x(x_i, y_i), z'_y(x_i, y_i), -1\right\}$ koordinatalarga ega bo‘ladi. U holda γ_i o‘tkir burchak $\vec{\boldsymbol{k}} = \{0,1,1\}$ va $\vec{\boldsymbol{n}}_i = \left\{z'_x(x_i, y_i), z'_y(x_i, y_i), -1\right\}$ vektorlar orasidagi burchak bo‘ladi va demak

$$\cos\gamma_i = \frac{\vec{\boldsymbol{k}}\vec{\boldsymbol{n}}_i}{|\vec{\boldsymbol{k}}||\vec{\boldsymbol{n}}_i|} = \frac{1}{\sqrt{1 + {z'_x}^2(x_i, y_i) + {z'_y}^2(x_i, y_i)}}$$

tenglikka ega bo‘lamiz. Bu tenglikni (9.42) tenglikka qo‘llasak

$$\Delta T_i = \sqrt{1 + {z'_x}^2(x_i, y_i) + {z'_y}^2(x_i, y_i)}\Delta D_i \tag{9.43}$$

tenglik hosil bo‘ladi.

(9.41) tenglikni inobatga olgan holda (9.43) formulani (9.39) formulaning o‘ng tomoniga qo‘yamiz, z_i ni esa $z(x_i, y_i)$ bilan almashtiramiz. S_i qismiy sirtlarning eng katta d diametri nolga intilganda D_i sohalarning ham diametrlari nolga intilishi ravshan. Shuning uchun bu almashtirishlardan song eng katta d diametrni nolga intiltirsak σ sirt bo‘yicha sirt integralini bu sirtning Oxy tekislikdagi proyeksiyasi bo‘yicha ikki o‘lchavli integral orqali ifodalovchi (9.40) tenglik hosil bo‘ladi.

Bu integrallardan birining mavjudligi ikkinchisining ham mavjudligini keltirib chiqaradi.◄

Agar σ sirt $y = y(x, z)$ yoki $x = x(y, z)$ tenglama bilan berilgan bo‘lsa, (9.40) formula mos ravishda

$$\iint_{\sigma} f(x, y, z)ds = \iint_{D_1} f(x, y(x, z), z) \cdot \sqrt{1 + {y'_x}^2 + {y'_z}^2}dxdz$$

va

$$\iint_{\sigma} f(x,y,z)ds = \iint_{D_2} f(x(y,z),y,z) \cdot \sqrt{1 + {x'_y}^2 + {x'_z}^2}dydz$$

ko‘rinishda bo‘ladi, bu yerda D_1 va D_2 sohalar σ sirtning mos ravishda Oxz va Oyz tekisliklardagi proyeksiyalari.

I tur sirt integralining xossalari. I tur sirt integrali quyidagi xossalarga ega:

1. O‘zgarmas ko‘paytuvchini integral belgisidan tashqariga chiqarish mumkin:

$$\iint_{\sigma} c \cdot f(x,y,z)ds = c \cdot \iint_{\sigma} f(x,y,z)ds.$$

2. Ikkita funksiya yig‘indisining sirt integrali bu funksiyalar sirt integrallarining yig‘indisiga teng:

$$\iint_{\sigma} (f(x,y,z) + g(x,y,z))ds = \iint_{\sigma} f(x,y,z)ds + \iint_{\sigma} g(x,y,z)ds.$$

3. Agar σ sirt faqat chegaralarida umumiy nuqtalarga ega bo‘ladigan va $\sigma = \sigma_1 \cup \sigma_2$ bo‘ladigan qilib ikkita σ_1 va σ_2 qismlarga ajratilsa,

$$\iint_{\sigma} f(x,y,z)ds = \iint_{\sigma_1} f(x,y,z)ds + \iint_{\sigma_2} f(x,y,z)ds$$

tenglik o‘rinli bo‘ladi.

4. Agar σ sirtda $f(x,y,z) \leq g(x,y,z)$ tengsizlik o‘rinli bo‘lsa,

$$\iint_{\sigma} f(x,y,z)ds \leq \iint_{\sigma} g(x,y,z)ds$$

bo‘ladi.

5. $\iint_{\sigma} ds = S$, bu yerda S berilgan σ sirtning yuzi.

6. $\left|\iint_{\sigma} f(x,y,z)ds\right| \leq \iint_{\sigma} |f(x,y,z)|ds.$

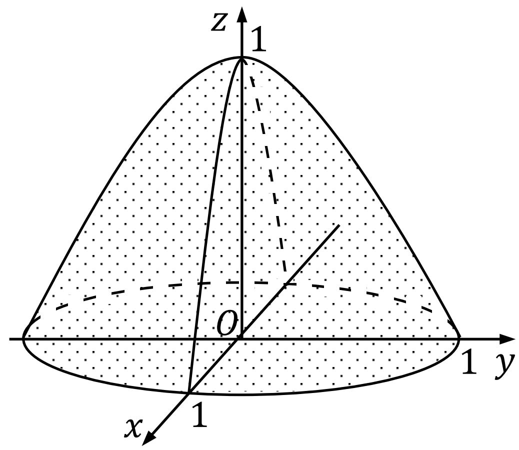
7. Agar $f(x,y,z)$ funksiya σ sirtda uzluksiz bo‘lsa, bu sirtda

$$\iint_{\sigma} f(x,y,z)ds = f(x_c,y_c,z_c) \cdot S$$

tenglikni qanoatlantiruvchi $C(x_c,y_c,z_c)$ nuqta mavjud (o‘rta qiymat haqidagi teorema).

Bu xossalarning isboti bevosita (9.40) tenglikdan va ikki o‘lchovli integralning xossalaridan kelib chiqadi. Shuning uchun biz ularni keltirib o‘tirmaymiz.

9.12-Misol. $\iint\limits_{\sigma} \sqrt{1+4x^2+4y^2}ds$ integralni hisoblang, bu yerda σ −sirt $z=1-x^2-y^2$ aylanma paraboloidning $z=0$ tekislik bilan kesib olingan qismi (9.15-rasm).



9.15-rasm

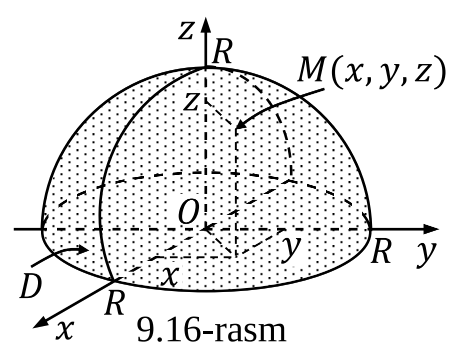
► $z=1-x^2-y^2$ tenglama bilan berilgan σ sirt Oxy tekislikka $x^2+y^2=1$ aylana bilan chegaralangan D sohaga proyeksiyalanadi (paraboloid tenglamasida $z=0$ deb olinsa aylana tenglamasi hosil bo‘ladi). Demak D soha $x^2+y^2\leq 1$ doiradan iborat ekan. Bu doirada $z=1-x^2-y^2$ funksiya va uning $z'_x=-2x$, $z'_y=$ $=-2y$ hosilalari uzluksiz. Shuning uchun (9.40) formulani qo‘llash mumkin:

$$\iint\limits_{\sigma} \sqrt{1+4x^2+4y^2}ds = \iint\limits_{D} \sqrt{1+4x^2+4y^2}\sqrt{1+4x^2+4y^2}dxdy =$$
$$= \iint\limits_{D} (1+4x^2+4y^2)dxdy.$$

So‘ngi integralda $x=r\cos\varphi$, $y=r\sin\varphi$ deb qutb koordinatalariga o‘tamiz:

$$\iint\limits_{D} (1+4x^2+4y^2)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 (1+4r^2)rdr =$$
$$= \int\limits_0^{2\pi} \left[\left(\frac{r^2}{2}+r^4\right)\Big|_0^1\right] d\varphi = \frac{3}{2}\int\limits_0^{2\pi} d\varphi = 3\pi. \ ◄$$

Agar σ sirt nuqtalarida biror massa $\mu=\mu(P)\geq 0$ sirt zichligi bilan taqsimlangan bo‘lsa, $\iint\limits_{\sigma} \mu(P)ds$ integralni bu *sirtning m massasi* deb talqin qilish mumkin.



9.16-rasm

9.13-Misol. Radiusi R ga teng bo‘lgan yarim sferaning $M(x,y,z)$ nuqtadagi sirt zichligi $\mu(M)=\sqrt{x^2+y^2}$ tenglik bilan aniqlansa, sirtning m massasini toping.

►9.16-rasmda radiusi R ga teng bo‘lgan $z=\sqrt{R^2-x^2-y^2}$ yarim sfera tasvirlangan. Bu yarim sferaning Oxy tekislikdagi proyeksiyasi $D: x^2+y^2\le R^2$ doiradan iborat bo‘ladi. Sirtning massasini (9.40) formulaga ko‘ra hisoblaymiz. Dastlab z'_x va z'_y hosilalarni topsak:

$$z'_x=-\frac{x}{\sqrt{R^2-x^2-y^2}},\qquad z'_x=-\frac{x}{\sqrt{R^2-x^2-y^2}}.$$

bo‘ladi. Bu hosilalarni (9.40) formulaga qo‘yib m massani topamiz:

$$m=\iint_\sigma \mu(P)ds=\iint_D \sqrt{x^2+y^2}\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}dxdy$$

$$=R\iint_D \frac{\sqrt{x^2+y^2}}{\sqrt{R^2-(x^2+y^2)}}dxdy.$$

So‘ngi integralda qutb koordinatalariga o‘tamiz:

$$\iint_D \frac{\sqrt{x^2+y^2}}{\sqrt{R^2-(x^2+y^2)}}dxdy=\int_0^{2\pi}d\varphi\int_0^R\frac{r}{\sqrt{R^2-r^2}}\cdot rdr=$$

$$=\int_0^{2\pi}d\varphi\cdot\int_0^R\frac{r^2dr}{\sqrt{R^2-r^2}}=$$

$$=\begin{vmatrix} r=R\sin t\,, dr=R\cos t\,dt \\ r=0, t=0, r=R, t=\pi/2\end{vmatrix}=2\pi\cdot\int_0^{\pi/2}\frac{R^2\sin^2 t\,R\cos t}{\sqrt{R^2-R^2\sin^2 t}}dt=$$

$$=2\pi R^2\int_0^{\pi/2}\sin^2 t\,dt=2\pi R^2\int_0^{\pi/2}\frac{1-\cos 2t}{2}dt=$$

$$=2\pi R^2\left(\frac{t}{2}-\frac{\sin 2t}{4}\right)\Big|_0^{\pi/2}=\frac{\pi^2R^2}{2}.$$

Demak,

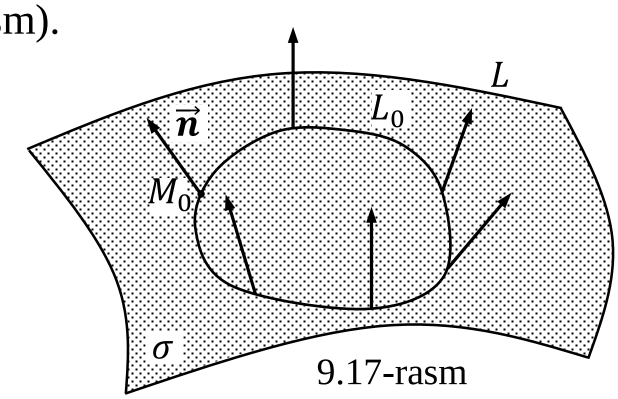
$$m=R\cdot\frac{\pi^2R^2}{2}=\frac{\pi^2R^3}{2}.$$ ◄

9.10. II tur sirt integrali

Bir tomonli va ikki tomonli sirtlar. Sirtning tomoni degan muhim tushunchani kiritamiz. Ba’zi hollarda bu tushunchani oson tasavvur qilish mumkin. Agar sirt $z=f(x,y)$ ko‘rinishdagi tenglama bilan berlgan bo‘lsa, uning yuqori va quyi qismi haqida gapirish mumkin. *Yopiq sirt* fazoda biror sohani chegaralab turadi va bu holda sirtning sohaga qaragan ichki qismi va tashqariga qaragan tashqi qismi bo‘ladi.

Ana shu intuitiv tasavvurlar asosida bir tomonli va ikki tomonli sirtlarga ta’rif beramiz.

L bo‘lakli silliq kontur bilan chegaralangan silliq σ sirtni qaraymiz. Demak sirning ixtiyoriy nuqtasida urinma tekislik mavjud va u urinish nuqtasi bilan birga uzluksiz o‘zgaradi (9.17-rasm).



9.17-rasm

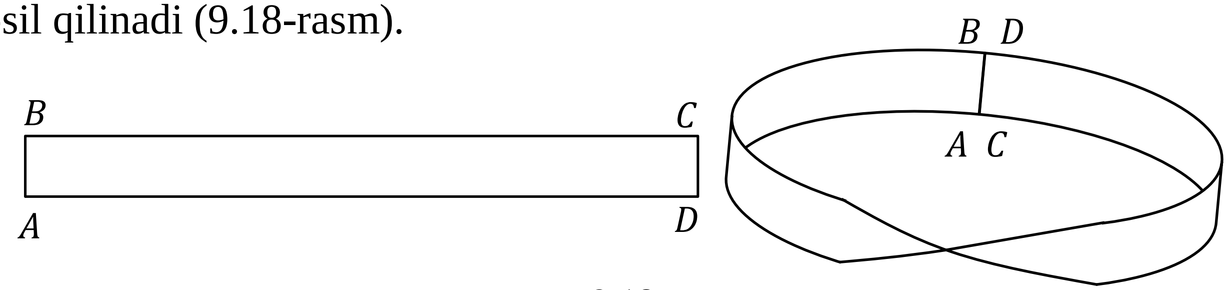
Sirtda birorta M_0 nuqta olamiz va bu nuqta orqali $\vec{\boldsymbol{n}}$ normal o‘tkazamiz, bunda normal uchun ikkita yo‘nalishdan biri tanlanadi (ular bir-biridan yo‘naltiruvchi kosinuslarining ishorasi bilan farqlanadi). Sirt bo‘ylab M_0 nuqtadan o‘tuvchi va L sirt chegarasi bilan kesishmaydigan uzluksiz L_0 yopiq kontur o‘tkazamiz. Boshlang‘ich holati M_0 nuqta bo‘ladigan M nuqtani L_0 yopiq kontur bo‘ylab uzluksiz harakatlantiramiz. Bunda M nuqtada o‘tkazilgan normal ham o‘z holatini uzluksiz o‘zgartiradi. M nuqta o‘zining M_0 boshlang‘ich holatga qaytganda ikki hol yuz berishi mumkin: yoki normal o‘zining boshlang‘ich holatdagi yo‘nalishiga qaytadi, yoki normal boshlang‘ich holatga qarama-qarshi yo‘nalishga qaytadi.

9.4-Ta’rif. Agar silliq σ sirtda yotuvchi M_0 nuqta orqali o‘tuvchi, sirt chegarasi bilan kesishmaydigan ixtiyoriy yopiq kontur bo‘ylab M nuqta M_0 boshlang‘ich holatdan chiqib yana shu M_0 boshlang‘ich holatga qaytganda birlik normal vektor o‘z yo‘nalishini o‘zgartirmasa, sirt *ikki tomonli* yoki *oriyentirlangan sirt* deb ataladi.

9.5-Ta’rif. Agar silliq σ sirtda yotuvchi M_0 nuqta orqali o‘tuvchi sirt chegarasi bilan kesishmaydigan birorta yopiq kontur bo‘ylab M nuqta M_0 boshlang‘ich holatdan chiqib yana shu M_0 boshlang‘ich holatga qaytganda birlik normal vektor yo‘nalishi qarama-qarshi tomonga o‘zgarsa, sirt *bir tomonli* yoki *oriyentirlanmagan sirt* deb ataladi.

Ikki tomonli sirtga misol sifatida tekislikni, ellipsoidni, umuman olganda $z = f(x, y)$ tenglama bilan berilgan ixtiyoriy sirtni keltirish mumkin, bu yerda $f(x, y)$, f'_x, f'_y, funksiyalar tekislikning biror D sohasida uzluksiz funksiyalar.

Bir tomonli sirtga Myobius lentasi deb ataluvchi sirtni misol keltirish mumkin. Bu sirt $ABCD$ to‘gri to‘rtburchak shakldagi lentani A nuqta C nuqta bilan, B nuqta esa D nuqta bilan ustma-ust tushadigan qilib yelimlash orqali hosil qilinadi (9.18-rasm).



9.18-rasm

Shunday qilib, ikki tomonli sirtda uzluksiz birlik normal vektorni ikki usul bilan tanlash mumkin ekan. Har bir nuqtada birlik normal vektorni uzluksiz o‘zgaradigan qilib tanlash orqali, sirtning tomonini aniqlagan bo‘lamiz.

II tur sirt integrali. $z = z(x, y)$ tenglama bilan berilgan σ ikki tomonli silliq sirtda $R(x, y, z)$ funksiya aniqlangan bo‘lsin. Sirtning ikki tomonidan birini, ya’ni sirt nuqtalaridagi normal yo‘nalishlarining birini tanlaymiz. Agar normal Oz o‘qi bilan o‘tkir burchak tashkil qilsa, sirtning *yuqori qismi*, o‘tmas burchak tashkil qilsa, sirtning *quyi tomoni* tanlangan deyiladi. σ sirtning tanlangan tomonini n ta $\sigma_1, \sigma_2, \dots, \sigma_n$ elementar sohalarga ajratamiz. σ_i qismiy sirtning Oxy tekislikdagi proyeksiyasini D_i orqali belgilaymiz. Har bir σ_i qismiy sirtda ixtiyoriy ravishda $M_i(x_i, y_i, z_i)$ nuqtani tanlaymiz va

$$\sum_{i=1}^{n} R(x_i, y_i, z_i)\Delta s_i \qquad (9.44)$$

yig‘indini tuzamiz, bu yerda Δs_i yuqorida zikr etilgan D_i sohaning yuzi bo‘lib, agar sirtning yuqori tomoni tanlangan bo‘lsa u “plyus” ishora bilan, quyi qismi tanlangan bo‘lsa “minus” ishora bilan olinadi. (9.44) yig‘indi $R(x, y, z)$ funksiyaning σ sirt bo‘yicha *integral yig‘indisi* deb ataladi.

σ_i elementar sohalar d_i diametrlarining eng kattasini λ orqali belgilaymiz: $\lambda = \max\limits_{1 \le i \le n} d_i$.

9.6-Ta’rif. Agar λ eng katta diametr nolga intilganda (9.44) integral yig‘indi chekli limitga intilsa va bu limit σ sirtni σ_i qismlarga bo‘lish usuliga va bu qismlardagi nuqtalarning tanlanishiga bog‘liq bo‘lmasa, u $R(x, y, z)$ funksiyaning σ sirt tanlangan tomoni bo‘yicha *II tur sirt integrali* deb ataladi va

$$I=\iint\limits_{\sigma} R(M)dxdy=\iint\limits_{\sigma} R(x,y,z)dxdy$$

orqali belgilanadi. Bu holda $R(x,y,z)$ funksiya x va y o‘zgaruvchilar bo‘yicha σ sirt bo‘yicha *integrallanuvchi* deyiladi.

Shunday qilib, ta’rifga ko‘ra

$$\iint\limits_{\sigma} R(x,y,z)dxdy=\lim_{\lambda\to 0}\sum_{i=1}^{n} R(x_i,y_i,z_i)\Delta s_i \qquad (9.45)$$

σ sirt tanlangan tomoni bo‘yicha $P(x,y,z)$ funksiyaning y va z o‘zgaruvchilar bo‘yicha

$$\iint\limits_{\sigma} P(x,y,z)dydz$$

va $Q(x,y,z)$ funksiyaning z va x o‘zgaruvchilar bo‘yicha

$$\iint\limits_{\sigma} Q(x,y,z)dzdx$$

II tur sirt integrallari xuddi shu singari aniqlanadi.

$$\iint\limits_{\sigma} P(x,y,z)dydz+\iint\limits_{\sigma} Q(x,y,z)dzdx+\iint\limits_{\sigma} R(x,y,z)dxdy \qquad (9.46)$$

yig‘indi umumlashgan II tur sirt integrali deb ataladi va

$$\iint\limits_{\sigma} P(x,y,z)dydz+Q(x,y,z)dzdx+R(x,y,z)dxdy \qquad (9.47)$$

orqali belgilanadi.

II tur sirt integralining xossalari. II tur sirt integralining xossalarini funksiyaga mos keluvchi hadi misolida qarab chiqamiz.

1) Sirt tomoni o‘zgarganda integral o‘z ishorasini o‘gartiradi.

2) O‘zgarmas ko‘paytuvchini integral belgisidan tashqariga chiqarish mumkin:

$$\iint\limits_{\sigma} c\cdot R(x,y,z)dxdy=c\cdot\iint\limits_{\sigma} R(x,y,z)dxdy$$

3) Ikkita integrallanuvchi funksiya yig‘indisining II tur sirt integrali bu funksiyalar mos integrallarining yig‘indisiga teng:

$$\iint\limits_{\sigma}[R_1(x,y,z)+R_2(x,y,z)]dxdy$$
$$=\iint\limits_{\sigma} R_1(x,y,z)dxdy+\iint\limits_{\sigma} R_2(x,y,z)dxdy$$

4) Agar σ sirt umumiy ichki nuqtalarga ega bo‘lmagan va $\sigma=\sigma_1\cup\sigma_2$ bo‘ladigan qilib ikkita σ_1 va σ_2 qismlarga ajratilgan bo‘lsa,

$$\iint\limits_{\sigma} R(x,y,z)dxdy=\iint\limits_{\sigma_1} R(x,y,z)dxdy+\iint\limits_{\sigma_2} R(x,y,z)dxdy$$

bo‘ladi.

5) Oz o‘qiga parallel bo‘lgan ixtiyoriy σ silindrik sirt bo‘yicha II tur sirt integrali nolga teng:

$$\iint_{\sigma} R(x,y,z)dxdy = 0.$$

II tur sirt integralini hisoblash. II tur sirt integralini hisoblash ikki o‘lchovli integralni hisoblashga keltiriladi.

$R(x,y,z)$ funksiya $z = z(x,y)$ tenglama bilan berilgan σ silliq sirtning barcha nuqtalarida uzluksiz, bu yerda $z(x,y)$ berilgan σ sirtning Oxy tekislikdagi proyeksiyasi bo‘lgan D_{xy} sohada uzluksiz funksiya.

σ sirtning nuqtalaridagi normallar Oz o‘q bilan o‘tkir burchak tashkil qiladigan tomonini olamiz. U holda $\Delta s_i > 0$ $(i = 1,2,\dots,n)$ bo‘ladi.

$z_i = z(x_i,y_i)$ bo‘lganligi uchun (9.44) yig‘indi

$$\sum_{i=1}^{n} R(x_i,y_i,z_i)\Delta s_i = \sum_{i=1}^{n} R(x_i,y_i,z(x_i,y_i))\Delta s_i \tag{9.48}$$

ko‘rinishni oladi. Bu tenglikning o‘ng tomonidagi yig‘indi uzluksiz $R(x,y,z(x,y))$ funksiya uchun D soha bo‘yicha integral yig‘indi bo‘ladi. (9.48) tenglikda σ_i elementar sohalar d_i diametrlarining λ eng kattasi nolga intilganda limitga o‘tsak,

$$\iint_{\sigma} R(x,y,z)dxdy = \iint_{D_{xy}} R(x,y,z(x,y))dxdy \tag{9.49}$$

x va y o‘zgaruvchilar bo‘yicha II tur sirt integralini ikki o‘lchovli integral orqali ifodalovchi formulani hosil qilamiz. Agar sirtning ikkinchi tomonini, ya’ni quyi tomoni tanlansa ikki o‘lchovli integral “minus” ishora bilan olinadi. Shuning uchun

$$\iint_{\sigma} R(x,y,z)dxdy = \pm\iint_{D_{xy}} R(x,y,z(x,y))dxdy \tag{9.50}$$

Xuddi shu singari

$$\iint_{\sigma} P(x,y,z)dydz = \pm\iint_{D_{yz}} P(x(y,z),y,z)dydz, \tag{9.51}$$

$$\iint_{\sigma} Q(x,y,z)dzdx = \pm\iint_{D_{zx}} Q(x,y(x,z),z)dzdx, \tag{9.52}$$

tengliklarni yozish mumkin, bu yerda D_{yz} va D_{zx} sohalar σ sirtning mos ravishda Oyz va Oxz tekisliklardagi proyeksiyalari.

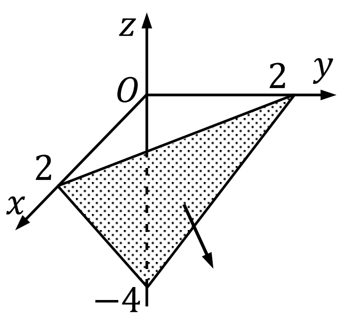
(9.51) formulada σ sirt $x = x(y,z)$, (9.52) formulada esa $y = y(x,z)$ tenglama bilan berilgan. Integrallar oldidagi ishoralar bu yerda ham σ sirt tomonining tanlanishiga qarab qo‘yiladi ((9.50) formuladagi singari sirt normali Ox (Oy) o‘q bilan o‘tkir burchak tashkil qilsa “plyus”, o‘tmas burchak tashkil qilsa “minus” ishora qo‘yiladi).

(9.47) umulashgan II tur sirt integralini hisoblash uchun σ sirtni uchala koordinata tekisliklariga proyeksiyalab, (9.50)-(9.52) formulalar yordamida hisoblanadi:

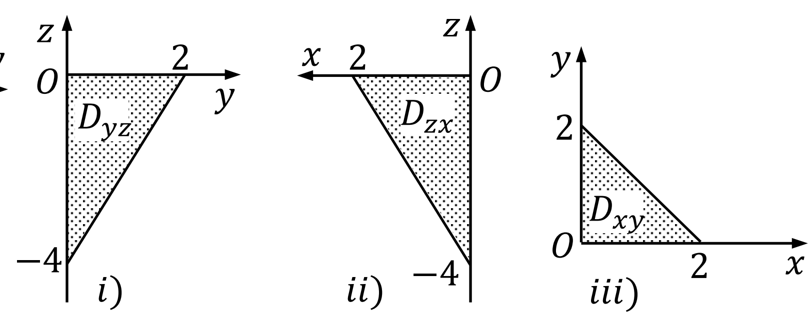
$$\iint\limits_{\sigma} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy =$$
$$= \pm \iint\limits_{D_{yz}} P(x(y,z),y,z)dydz \pm$$
$$\pm \iint\limits_{D_{zx}} Q(x,y(x,z),z)dzdx \pm \iint\limits_{D_{xy}} R(x,y,z(x,y))dxdy$$

Agar σ yopiq sirt bo‘lsa, sirtning tashqi tomoni bo‘yicha II tur sirt integrali $\oiint\limits_{\sigma}$ orqali, ichki tomoni bo‘yicha esa $\oiint\limits_{\sigma-}$ orqali belgilanadi.

9.14-Misol. $\iint\limits_{\sigma} 2xdydz - 2zdzdx + xydxdy$ integralni hisoblang, bu yerda σ sirt $2x + 2y - z = 4$ tekislikning V oktantdagi bo‘lagining yuqori qismi.



9.19-rasm



9.20-rasm

► 9.19-rasmda tekislikning berilgan qismi tasvirlangan. Bu tekislikning normali vektordan iborat bo‘ladi va uning yo‘naltiruvchi kosinuslari

$$\cos\alpha = \frac{2}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{2}{3} > 0, \cos\beta = \frac{2}{3} > 0,$$
$$\cos\gamma = -\frac{1}{3} < 0$$

bo‘ladi. Demak tekislik ko‘rsatilgan tomonining normali Ox va Oy o‘qlar bilan o‘tkir, Oz o‘q bilan esa o‘tmas burchak tashkil qiladi. Shuning uchun (9.51) va (9.52) formulalardagi ikki o‘lchovli integral oldiga “plyus”, (9.50) formulada esa “minus” ishora qo‘yamiz. Tekislik qaralayotgan qismining tekisliklardagi proyeksiyalari 9.20-rasmlarda tasvirlangan:

Endi har bir integralni alohida-alohida hisoblaymiz:

$$I_1 = \iint_\sigma 2xdydz = \iint_{D_{yz}} (4-2y+z)dydz = \int_0^2 dy \int_0^{2y-4} (4-2y+z)dz =$$

$$= \int_0^2 \left[\left((4-2y)z + \frac{1}{2}z^2\right)\Bigg|_0^{2y-4}\right]dy = \frac{3}{2}\int_0^2 (4-2y)^2 dy =$$

$$= -\frac{1}{4}(4-2y)^3\Bigg|_0^2 = 16;$$

$$I_2 = \iint_\sigma -2zdzdx = \iint_{D_{zx}} (-2z)dzdx = -\int_0^2 dx \int_0^{2x-4} 2zdz =$$

$$= -\int_0^2 [z^2|_0^{2x-4}]dx = -\int_0^2 (2x-4)^2 dx = -\frac{1}{6}(2x-4)^3\Bigg|_0^2 = -\frac{32}{3};$$

$$I_3 = \iint_\sigma xydxdy = \iint_{D_{xy}} xydxdy = \int_0^2 dx \int_0^{2-x} xydy = \int_0^2 \left[\frac{1}{2}xy^2\Bigg|_0^{2-x}\right]dx =$$

$$= \int_0^2 \frac{1}{2}x(2-x)^2 dx = \frac{1}{2}\int_0^2 (4x-4x^2+x^3)dx =$$

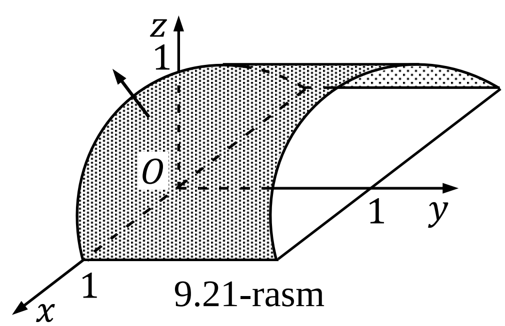
$$= \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Bigg|_0^2 = \frac{4}{3}.$$

Shunday qilib,

$$I = I_1 + I_2 + I_3 = 16 - \frac{32}{3} + \frac{4}{3} = 6\frac{2}{3}. \blacktriangleleft$$

9.15-Misol. $\iint_\sigma (y^2+z^2)dxdy$ integralni hisoblang, bu yerda σ sirt $z = \sqrt{1-x^2}$ sirtning $y=0$ va $y=1$ tekisliklar bilan ajratilgan qismining yuqori tomoni (9.21-rasm).

►Mazkur sirtning Oxy tekislikdagi proyeksiyasi $0 \le y \le 1$, $-1 \le x \le 1$ tengsizliklar bilan aniqlangan to‘g‘ri to‘rtburchakdan iborat. U holda berilgan integralni (9.50) formulaga ko‘ra hisoblaymiz:



9.21-rasm

$$\iint_\sigma (y^2+z^2)dxdy = \iint_{D_{xy}} (y^2 + (\sqrt{1-x^2})^2)dxdy =$$

$$= \int_{-1}^{1} dx \int_{0}^{1} (y^2 + 1 - x^2) dy = \int_{-1}^{1} \left(\frac{y^3}{3} + y - x^2 y \right) \Bigg|_0^1 dx =$$

$$= \int_{-1}^{1} \left(\frac{4}{3} - x^2 \right) dx = \left(\frac{4}{3} - \frac{x^3}{3} \right) \Bigg|_{-1}^{1} = 2. \blacktriangleleft$$

9.16-Misol. $\oiint\limits_{\sigma} xdydz + ydzdx + zdxdy$ integralni hisoblang, bu yerda σ sirt $x^2 + y^2 + z^2 = 1$ sferaning tashqi tomoni.

►Integrallash sirtining uchala koordinata o‘qlariga va tekisliklariga nisbatan simmetrikligi tufayli va integral ostidagi ifodaning ko‘rinishidan kelib chiqib

$$\oiint\limits_{\sigma} xdydz + ydzdx + zdxdy = 3 \oiint\limits_{\sigma} zdxdy$$

deb olishimiz mumkin. Shuning uchun bitta integralni hisoblash bilan chegaralanamiz. Songi integralni ikkita yarim sharlarning tashqi tomonlari

$$\oiint\limits_{\sigma} zdxdy = \iint\limits_{\sigma_1} zdxdy + \iint\limits_{\sigma_2} zdxdy$$

($\sigma_1: z = \sqrt{1 - x^2 - y^2}$ yuqori va $\sigma_2: z = -\sqrt{1 - x^2 - y^2}$ quyi yarim shar) bo‘yicha integrallar yig‘indisi shaklida ifodalash mumkin. Yuqori yarim sharning tashqi tomoni uchun $\cos\gamma > 0$ va demak, (9.50) formulada “plyus” ishora olinadi. Quyi yarim shar uchun $\cos\gamma < 0$ va demak, (9.50) formulada “minus” ishora olinadi, quyi yarim shar tenglamasidagi “minus” ishoraning hisobiga quyi yarim shar boyicha integral yuqori yarim shar bo‘yicha integral bilan ham qiymati ham ishorasi boyicha tenglashadi. Shuning uchun $\iint\limits_{\sigma} zdxdy = 2 \iint\limits_{\sigma_1} zdxdy$ bo‘ladi, bu yerda integrallash yuqori yarim sharning tashqi tomoni bo‘yicha olinmaoqda. Yarim sharning Oxy tekislikdagi proyeksiyasi $D_{xy} = \{(x, y): x^2 + y^2 \leq 1\}$ doiradan iborat bo‘ladi. U holda

$$\iint\limits_{\sigma_1} zdxdy = \iint\limits_{D_{xy}} \sqrt{1 - x^2 - y^2} dxdy$$

Ikki o‘lchovli integralda qutb koordinatalariga o‘tsak

$$\iint\limits_{D_{xy}} \sqrt{1 - x^2 - y^2} dxdy = \int_{0}^{2\pi} d\varphi \int_{0}^{1} \sqrt{1 - r^2} \cdot rdr =$$

$$= \int_{0}^{2\pi} \left[- \frac{(1 - r^2)^{3/2}}{3} \Bigg|_0^1 \right] d\varphi = \frac{1}{3} \int_{0}^{2\pi} d\varphi = \frac{2}{3}\pi.$$

Demak,

$$\oiint_{\sigma} zdxdy = 2 \cdot \frac{2}{3}\pi = \frac{4}{3}\pi,$$

bundan esa

$$\oiint_{\sigma} xdydz + ydzdx + zdxdy = 3 \cdot \frac{4}{3}\pi = 4\pi \quad \blacktriangleleft$$

I va II tur sirt integrallari orasidagi bog‘liqlik. II tur sirt integralini boshqa usul bilan ham keltirib chiqarish mumkin, aniqrog‘i, integral ostida maxsus ifodalar turgan I tur sirt integralidan hosil qilinadi. Orieyentirlangan σ sirtning ixtiyoriy nuqtasidagi normalning yo‘naltiruvchi kosinuslarini $\cos\alpha$, $\cos\beta$ va $\cos\gamma$ orqali belgilaymiz. U holda $\Delta s_i \approx \cos\gamma_i \, \Delta S_i$ taqribiy tenglikni yozish mumkin, bu yerda Δs_i yuza σ_i qismiy sirtining yuzi, ΔS_i esa $\sigma_i$qismiy sirtining Oxy tekislikdagi D_i proyeksiyasining yuzi, γ_i burchak σ sirtga biror $M_i \in \sigma_i$ nuqtada o‘tkazilgan normal bilan Oz o‘q tashkil qilgan burchak. Shuning uchun bu taqribiy tenglik o‘rniga

$$dxdy = \cos\gamma \cdot ds \tag{9.53}$$

tenglikni yozish mumkin. Xuddi shu singari

$$dxdz = \cos\beta \cdot ds, \; dydz = \cos\alpha \cdot ds \tag{9.54}$$

tengliklar o‘rinli bo‘ladi.

(9.53), (9.54) formulalarni II tur sirt integraliga qo‘llaymiz, natijada:

1) $R(x,y,z)$ funksiyaning Oxy tekislik bo‘yicha II tur sirt integrali I tur sirt integrali orqali quyidagi formula yordamida ifodalanadi:

$$\iint_{\sigma} R(x,y,z)dxdy = \iint_{\sigma} R(x,y,z)\cos\gamma \, ds \tag{9.55}$$

2) $Q(x,y,z)$ funksiyaning Oxz tekislik bo‘yicha II tur sirt integrali I tur sirt integrali orqali quyidagi formula yordamida ifodalanadi:

$$\iint_{\sigma} Q(x,y,z)dzdx = \iint_{\sigma} Q(x,y,z)\cos\beta \, ds \tag{9.56}$$

3) $P(x,y,z)$ funksiyaning Oyz tekislik bo‘yicha II tur sirt integrali I tur sirt integrali orqali quyidagi formula yordamida ifodalanadi:

$$\iint_{\sigma} P(x,y,z)dydz = \iint_{\sigma} P(x,y,z)\cos\alpha \, ds \tag{9.57}$$

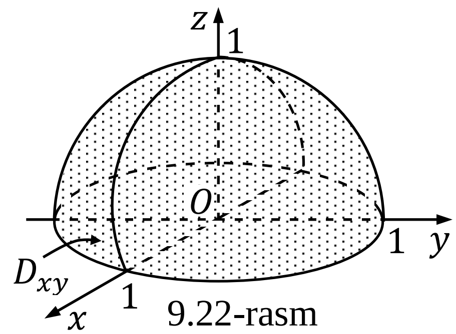
(9.55)-(9.57) formulalarni hadma-had qo‘shib sirtning tanlangan tomoni bo‘yicha II tur sirt integralini I tur sirt integrali bo‘yicha ifodalovchi

$$\iint_{\sigma} Pdydz + Qdzdx + Rdxdy =$$

$$= \iint_{\sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds \tag{9.58}$$

formulani hosil qilamiz.

9.17-Misol. $\iint\limits_{\sigma} z\cos\gamma\, ds$ integralni hisoblang, bu yerda σ sirt $x^2 + y^2 + z^2 = 1$ sferaning Oxy tekislikdan yuqorida joylashgan qismining tashqi tomoni, γ esa σ sirt normalining Oz o‘qi bilan tashkil qilgan o‘tkir burchagi (9.22-rasm).



9.22-rasm

►Sirt integrallarini bog‘lovchi (9.53) formulaga ko‘ra

$$\iint\limits_{\sigma} z\cos\gamma\, ds = \iint\limits_{\sigma} zdxdy$$

bo‘ladi. σ sirtning Oxy tekislikdagi D_{xy} proyeksiyasi $D_{xy} = \{(x, y): x^2 + y^2 \leq 1\}$ doiradan iborat. Shuning uchun (9.50) formulaga ko‘ra

$$\iint\limits_{\sigma} zdxdy = \iint\limits_{D_{xy}} \sqrt{1 - x^2 - y^2}dxdy$$

tenglikka ega bo‘lamiz. O‘ng tomondagi ikki o‘lchovli integralda qutb koordinatalariga o‘tamiz:

$$\iint\limits_{D_{xy}} \sqrt{1 - x^2 - y^2}dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \sqrt{1 - r^2} \cdot rdr =$$

$$= \int\limits_0^{2\pi} \left[-\frac{(1 - r^2)^{3/2}}{3} \Bigg|_0^1 \right] d\varphi = \frac{1}{3}\int\limits_0^{2\pi} d\varphi = \frac{2}{3}\pi. \quad ◄$$

II tur sirt integralining fizik ma’nosi. Bir birlik vaqt mobaynida berilgan σ sirt orqali sizib o‘tadigan suyuqlik miqdorini (ba’zan bu holda σ sirt orqali suyuqlik sarfi ham deb yuritiladi) topish masalasini qaraymiz. Suyuqlikning zichligi o‘zgarmas, suyuqlik sirt orqali sizib o‘ta oladi va suyuqlikning oqishi barqaror, ya’ni M nuqtadagi tezlik vektori $\vec{v}(M)$ vaqt bo‘yicha o‘zgarmas deb faraz qilamiz.

Agar σ sirt yuzi S ga teng bo‘lgan tekislik qismi, $\vec{v}(M)$ vektor esa bu tekislikka perpendikulyar bo‘lsa, u holda bir birlik vaqt mobaynida σ sirt orqali sizib o‘tgan suyuqlik hajmi $V = |\vec{v}|S$ ga teng bo‘ladi. Agar $\vec{v}(M)$ vektor tekislik normali bilan φ burchak tashkil qilsa, $V = |\vec{v}|S\cos\varphi$ bo‘ladi. Tekislik normali vektorining yo‘nalishiga qarab suyuqlik sarfi musbat, manfiy, xususiy holda nol ham bo‘lishi ravshan.

σ silliq sirt va uning ixtiyoriy $M(x, y, z) \in \sigma$ nuqtasida $\vec{v}(M) = \vec{v}(x, y, z)$ vektor funksiya yordamida tezlik vektori aniqlangan bo‘lsin. σ sirtni yuzlari Δs_i va diametrlari d_i $(i = 1,2, \dots, n)$ bo‘lgan n ta $\sigma_1, \sigma_2, \dots, \sigma_n$ qismlarga bo‘lamiz va har bir qismiy sirtda ixtiyoriy $M_i(x_i, y_i, z_i)$ nuqtani tanlaymiz. d_i diametrlarning kichik qiymatlarida σ_i qismiy sirtni uning σ sirtga M_i nuqtada o‘tkazilgan *urinma tekislikdagi proyeksiyasi* bilan almashtirish mumkinligi ravshan. Bundan tashqari qismiy sirtda suyuqlik tezligi vektorini o‘zgarmas va $\vec{v}(M_i)$ ga teng deb faraz qilamiz. Bu qilingan farazlarda sirtga M_i nuqtada o‘tkazilgan normalning $\vec{n}(M_i)$ birlik vektori yo‘nalishida σ_i qismiy sirt orqali suyuqlik sarfining umumiy hajmi taqriban

$$V_i \approx |\vec{v}(M_i)| \cos\varphi_i \, \Delta s_i$$

qiymatga teng bo‘ladi, bu yerda φ_i suyuqlik tezligi $\vec{v}(M_i)$ vektori bilan, $\vec{n}(M_i)$ birlik vektori orasidagi burchak. Butun sirt orqali suyuqlik sarfining umumiy hajmi

$$V \approx \sum_{i=1}^{n} |\vec{v}(M_i)| \cos\varphi_i \, \Delta s_i \tag{9.59}$$

yig‘indi qiymatiga taqriban teng bo‘ladi.

σ sirtni σ_i qismiy sirtlarga bo‘lishlarda ularning o‘lchamlari qanchalik kichik bo‘lsa, so‘ngi tenglikdagi xatolik shunchalik kichik bo‘lishi ravshan. Shuning uchun,

$$V = \lim_{\lambda \to 0} \sum_{i=1}^{n} |\vec{v}(M_i)| \cos\varphi_i \, \Delta s_i \tag{9.60}$$

deb olish mumkin, bu yerda $\lambda = \max\limits_{1 \le i \le n} d_i$.

$\cos\alpha(M_i)$, $\cos\beta(M_i)$, $\cos\gamma(M_i)$ birlik $\vec{n}(M_i)$ vektorning yo‘naltiruvchi kosinuslari bo‘lsin, ya’ni

$$\vec{n}(M_i) = \cos\alpha(M_i)\,\vec{i} + \cos\beta(M_i)\,\vec{j} + \cos\gamma(M_i)\,\vec{k}$$

va $P(M), Q(M), R(M)$ lar esa $\vec{v}(M)$ suyuqlik tezligi vektorining koordinatalari bo‘lsin:

$$\vec{v}(M) = P(M)\vec{i} + Q(M)\vec{j} + R(M)\vec{k} \tag{9.61}$$

U holda

$$|\vec{v}(M_i)| \cos\varphi_i = \boldsymbol{v}(M_i)\boldsymbol{n}(M_i) = P(M_i)\cos\alpha(M_i) +$$
$$+Q(M_i)\cos\beta(M_i) + R(M_i)\cos\gamma(M_i)\,, \qquad M_i \in \sigma_i.$$

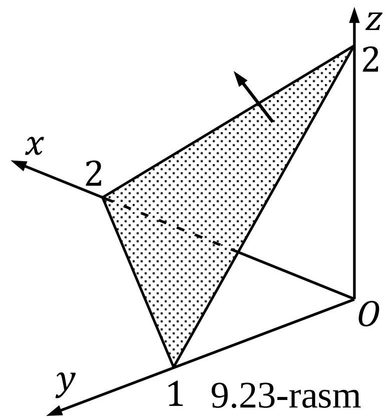
Bu tenglikni (9.60) tenglikka qo‘ysak, I tur sirt integralining ta’rifiga ko‘ra

$$V = \lim_{\lambda \to 0} \sum_{i=1}^{n} \left(P(M_i)\cos\alpha(M_i) + Q(M_i)\cos\beta(M_i) + R(M_i)\cos\gamma(M_i)\right)\Delta s_i$$
$$= \iint\limits_{\sigma} \left(P(M)\cos\alpha(M) + Q(M)\cos\beta(M) + R(M)\cos\gamma(M)\right)ds$$

tenglikni hosil qilamiz. Shuning uchun (9.58) formulani inobatga olsak, σ sirt orqali (9.61) ko‘rinishdagi tezlik vektoriga ega bo‘lgan suyuqlik oqimining vaqt birligidagi umumiy hajmi

$$V = \iint_{\sigma} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy \qquad (9.62)$$

II tur sirt integrali bilan hisoblanadi.



9.23-rasm

9.18-Misol. $x + 2y + z = 2$ tekislikning birinchi oktantdagi qismi bo‘ylab $\vec{v} = -x\vec{i} + 2y\vec{j} + z\vec{k}$ tezlik vektoriga ega bo‘lgan suyuqlik oqimining vaqt birligida oqib o‘tgan suyuqlik hajmini toping (tekislik normali Oz o‘q bialn o‘tkir burchak tashkil qiladi).

►Sirtning Oxy, Oyz va Ozx tekisliklardagi proyeksiyalari mos ravishda $D_{xy} = \left\{(x,y): 0 \le x \le 2, 0 \le y \le \frac{2-x}{2}\right\}$, $D_{yz} = \{(y,z): 0 \le y \le 1, 0 \le z \le 2 - 2y\}$, $D_{zx} = \{(x,z): \ 0 \le x \le 2, 0 \le z \le 2 - x\}$ uchburchaklardan iborat bo‘ladi (9.23-rasm). Oqib o‘tgan suyqlik hajmini (9.62) formulaga ko‘ra hisoblaymiz. Bu yerda $P(x,y,z) = -x$, $Q(x,y,z) = 2y$, $R(x,y,z) = z$ bo‘ladi.

(9.62) integral uchta integralning yig‘indisidan iborat va ularni alohida-alohida hisoblaymiz:

$$I_1 = \iint_{\sigma} -xdydz = \iint_{D_{yz}} (2y + z - 2)dydz = \int_0^1 dy \int_0^{2-2y} (2y + z - 2)dz =$$

$$= \int_0^1 \left[\left((2y-2)z + \frac{z^2}{2}\right)\Bigg|_0^{2-2y}\right] dy = \int_0^1 \left(-\frac{(2y-2)^2}{2}\right) dy =$$

$$= -\frac{(2y-2)^3}{12}\Bigg|_0^1 = -\frac{2}{3};$$

$$I_2 = \iint_{\sigma} 2ydzdx = \iint_{D_{zx}} (2 - x - z)dzdx = \int_0^2 dx \int_0^{2-x} (2 - x - z)dz =$$

$$= \int_0^2 \left[\left((2-x)z - \frac{z^2}{2}\right)\Bigg|_0^{2-x}\right] dx = \int_0^2 \frac{(2-x)^2}{2} dx = -\frac{(2-x)^3}{6}\Bigg|_0^2 = \frac{4}{3};$$

$$I_3 = \iint_{\sigma} z dx dy = \iint_{D_{xy}} (2 - x - 2y) dx dy = \int_0^2 dx \int_0^{(2-x)/2} (2 - x - 2y) dy =$$

$$= \int_0^2 \left[\left((2-x)y - y^2\right)\Big|_0^{(2-x)/2}\right] dx = \int_0^2 \frac{(2-x)^2}{4} dx = -\frac{(2-x)^3}{12}\Bigg|_0^2 = \frac{2}{3}.$$

Bu integrallarni qo‘shib, vaqt birligida sirt orqali oqib o‘tgan suyuqlik hajmini topamiz:

$$V = I_1 + I_2 + I_3 = -\frac{2}{3} + \frac{4}{3} + \frac{2}{3} = \frac{4}{3}. \quad ◄$$

9.11. Ostrogradskiy-Gauss formulasi

Yopiq sirt bo‘yicha II tur sirt integrali bilan bu sirt chegaralagan hajm boyicha olingan uch o‘lchovli integral orasidagi bog‘liqlikni quyidagi teorema o‘rnatadi:

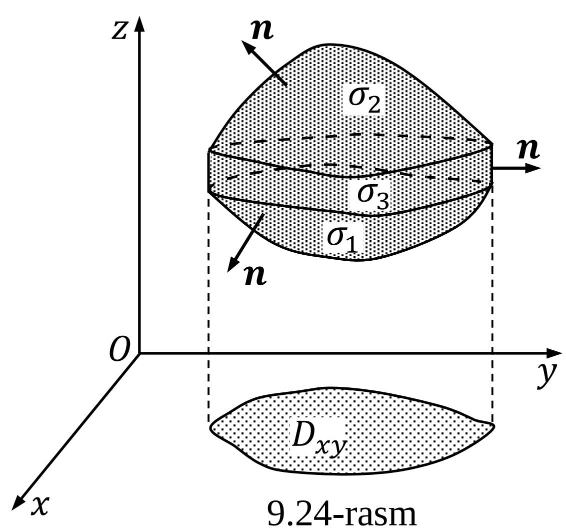
9.6-Teorema. Agar $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ funksiyalar va ularning xususiy hosilalari biror fazoviy V sohada uzluksiz bo‘lsa, quyidagi formula o‘rinli

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx dy dz = \oiint_{\sigma} P dy dz + Q dz dx + R dx dy \qquad (9.63)$$

bu yerda σ fazoviy V sohaning chegarasi va integrallash σ sirtning tashqi tomoni bo‘yicha olinadi.

(9.63) formula *Ostrogradskiy-Gauss* formulasi deb ataladi.

► V soha quyidan $z = z_1(x, y)$ tenglama bilan aniqlanadigan σ_1 sirt bilan, yuqoridan esa tenglamasi $z = z_2(x, y)$ bo‘lgan σ_2 sirt bilan, yon tomondan yasovchilari Oz o‘qqa parallel bo‘lgan σ_3 sirt bilan chegaralangan bo‘lsin, bu yerda $z_1(x, y)$ va $z_2(x, y)$ funksiyalar V sohaning Oxy tekislikdagi proyeksiyasi bo‘lgan D_{xy} sohada



9.24-rasm

uzluksiz va uning nuqtalarida $z_1(x, y) \leq z_2(x, y)$ deb faraz qilamiz (9.24-rasm).

$R(x, y, z)$ funksiyaga mos keluvchi uch o‘lchovli integralni qaraymiz:

$$\iiint_V \frac{\partial R}{\partial z} dxdydz =$$

$$= \iint_{D_{xy}} dxdy \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z} dz =$$

$$= \iint_{D_{xy}} R(x, y, z_2(x, y))dxdy - \iint_{D_{xy}} R(x, y, z_2(x, y))dxdy.$$

O‘ng tomondagi ikki o‘lchovli integrallarni (9.50) formulalarga ko‘ra σ_1 va σ_2 sirtlar bo‘yicha II tur sirt integrallari bilan almashtiramiz (σ_1 sirt normali Ox o‘q bilan o‘tmas burchak tashkil qilganlini sababli “minus” ishora olamiz). Natijada

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_{\sigma_2} Rdxdy + \iint_{\sigma_1} Rdxdy$$

tenglikka ega bo‘lamiz.

II sirt integrallarining 5) xossasiga ko‘ra σ_3 sirtning tashqi tomoni bo‘yicha olingan $\iint_{\sigma_3} Rdxdy$ integral nolga teng. Shuning uchun uni so‘ngi tenglikning o‘ng tomoniga qo‘shib

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_{\sigma_2} Rdxdy + \iint_{\sigma_1} Rdxdy + \iint_{\sigma_3} Rdxdy$$

tenglikni yozish mumkin. $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$ ekanligidan

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \oiint_\sigma Rdxdy \tag{9.64}$$

kelib chiqadi.

$$\iiint_V \frac{\partial Q}{\partial y} dxdydz = \oiint_\sigma Qdxdz \tag{9.65}$$

$$\iiint_V \frac{\partial P}{\partial x} dxdydz = \oiint_\sigma Pdydz \tag{9.66}$$

tengliklar ham xuddi shu singari isbotlanadi.

(9.64), (9.65) va (9.66) formulalarni hadma-had qo‘shib (9.63) Ostrogradskiy-Gauss formulasini hosil qilamiz.◄

9.19-Misol. $\oiint\limits_{\sigma} xdydz + 2ydzdx + 3zdxdy$ integralni hisoblang, bu yerda σ sirt $x + y + z = 2$, $x = 0$, $y = 0$, $z = 0$ tekisliklar bilan chegaralangan piramidaning tashqi tomoni.

► Ostrogradskiy-Gauss formulasiga ko‘ra

$$\oiint\limits_{\sigma} xdydz + 2ydzdx + 3zdxdy = \iiint\limits_{V} (1 + 2 + 3)dxdydz =$$

$$= 6\iiint\limits_{V} dxdydz = 6\int\limits_{0}^{2} dx \int\limits_{0}^{2-x} dy \int\limits_{0}^{2-x-y} dz =$$

$$= 6\int\limits_{0}^{2} dx \int\limits_{0}^{2-x} \left[z|_{0}^{2-x-y}\right]dy = 6\int\limits_{0}^{2} dx \int\limits_{0}^{2-x} (2 - x - y)dy =$$

$$= 6\int\limits_{0}^{2} \left[\left((2 - x)y - \frac{y^2}{2}\right)\Bigg|_{0}^{2-x}\right]dx = 6\int\limits_{0}^{2} \frac{1}{2}(2 - x)^2 dx =$$

$$= 3\left(-\frac{(2 - x)^3}{3}\Bigg|_{0}^{2}\right) = 8.$$

qiymatni hosil qildik.◄

9.20-Misol. $\oiint\limits_{\sigma} x^3dydz + y^3dzdx + z^3dxdy$ integralni hisoblang, bu yerda σ sirt $x^2 + y^2 + z^2 = R^2$ sferaning tashqi tomoni.

► Ostrogradskiy-Gauss formulasiga ko‘ra

$$\oiint\limits_{\sigma} x^3dydz + y^3dzdx + z^3dxdy = 3\iiint\limits_{V} (x^2 + y^2 + z^2)dxdydz,$$

uch o‘lchovli integralda sferik koordinatalrni kiritib

$$3\iiint\limits_{V} (x^2 + y^2 + z^2)dxdydz = 3\int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{\pi} \sin\theta \, d\theta \int\limits_{0}^{R} \rho^4 d\rho = \frac{12\pi\rho^5}{5}$$

qiymatga ega bo‘lamiz.◄

9.12. Stoks formulasi

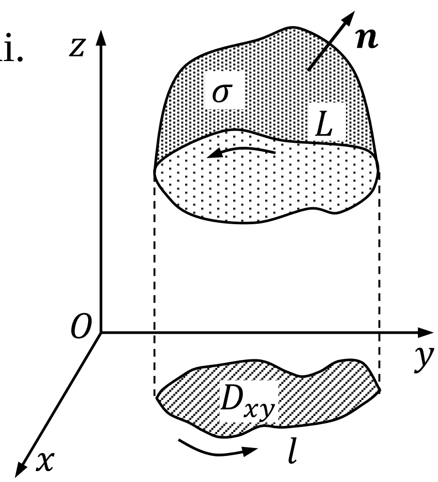
Stoks formulasi sirt va egri chiziqli integrallar o‘rtasida aloqa o‘rnatadi. Ostrogradskiy-Grin, Ostrogradskiy-Gauss formulalari singari Stoks formulasi ham ko‘p radbiqlarga ega.

9.7-Teorema. Agar $P(x, y, z)$, $Q(x, y, z)$ va $R(x, y, z)$ funksiyalar va ularning birinchi tartibli xususiy hosilalari oriyentirlangan σ sirt nuqtalarida uzluksiz bo‘lsa,

$$\iint_{\sigma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx =$$
$$= \oint_{L} Pdx + Qdy + Rdz \qquad (9.67)$$

tenglik orinli bo‘ladi, bu yerda chap tomondagi II tur sirt integralida integrallash σ sirtning yuqori qismi bo‘yicha, $L -$ egri chiziq σ sirtning chegarasi va integrallash musbat yo‘nalishda olib boriladi (ya’ni L egri chiziqni aylanib chiqish soat strelkasi harakatiga qarama-qarshi yo‘nalishda bo‘ladi).

(9.67) formula *Stoks formulasi* deb ataladi.



9.25-rasm

► σ sirt $z = z(x, y)$ tenglama bilan berilgan bo‘lsin, $z(x, y)$, $z'_x(x, y)$, $z'_y(x, y)$ funksiyalar σ sirtning Oxy tekislikdagi proyeksiyasi bo‘lgan D_{xy} sohada uzluksiz va D_{xy} sohaning chegarasi l egri chiziq bo‘lsin. Oz o‘qiga parallel bo‘lgan har qanday to‘g‘ri chiziq σ sirtni ko‘pi bilan bitta nuqtada kesib o‘tadi deb faraz qilamiz (9.25-rasm). σ sirtning yuqori tomonini tanlaymiz. Dastlab $\oint_L P(x, y, z)dx$ integralni qaraymiz.

$P(x, y, z)$ funksiyaning L egri chiziq nuqtalaridagi qiymati $P(x, y, z(x, y))$ funksiyaning l egri chiziq nuqtalaridagi qiymatiga teng. Shuning uchun

$$\oint_L P(x, y, z)dx = \oint_l P(x, y, z(x, y))dx.$$

O‘ng tomondagi integralga Ostrogradskiy-Grin formulasini qo‘llab

$$\oint_l P(x, y, z(x, y))dx = \iint_{D_{xy}} \left(0 - \frac{\partial}{\partial y} P(x, y, z(x, y)\right) dxdy =$$
$$= -\iint_{D_{xy}} \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial y}\right) dxdy \qquad (9.68)$$

tenglikka ega bo‘lamiz. Bu yerda integral ostidagi funksiya $P(x, y, z)$ funksiyadan z o‘rniga $z(x, y)$ qo‘yilgandan song y bo‘yicha xususiy hosila olish orqali hosil qilingan.

σ sirtning yuqori tomoni bo'lganligi uchun (ya'ni $\cos\gamma > 0$), normal $\{-z'_x, -z'_y, 1\}$ koordinatalarga ega bo'ladi. Ma'lumki yo'naltiruvchi kosinuslar mos koordinatalarga proporsional, shuning uchun

$$\frac{\cos\beta}{\cos\gamma} = \frac{-z'_y}{1} = -z'_y.$$

Bu tenglikni (9.68) tenglikning o'ng tomoniga qo'llab

$$\oint_l P(x,y,z(x,y))dx = -\iint_{D_{xy}} \left(\frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)dxdy =$$

$$= -\iint_{D_{xy}} \left(\frac{\partial P}{\partial y}\cos\gamma - \frac{\partial P}{\partial z}\cos\beta\right)\frac{1}{\cos\gamma}dxdy$$

tenglikka ega bo'lamiz. Agar (9.50), (9.53) va (9.55) formulalarni inobatga olsak

$$\oint_l P(x,y,z(x,y))dx = -\iint_\sigma \left(\frac{\partial P}{\partial y}\cos\gamma - \frac{\partial P}{\partial z}\cos\beta\right)ds$$

tenglik hosil bo'ladi. O'ng tomondagi integral I tur sirt integralidan iborat, unga yana (9.55) va (9.56) formulalarni qo'llab

$$\oint_L P(x,y,z)dx = -\iint_\sigma \frac{\partial P}{\partial y}dxdy + \iint_\sigma \frac{\partial P}{\partial z}dzdx \qquad (9.69)$$

II tur egri chiziqli integralni II tur sirt integrali orqali ifodaladik.

Xuddi shu singari (9.67) tenglikning o'ng tomonidagi $\oint_L Qdy$, $\oint_L Rdz$ integrallar uchun

$$\oint_L Q(x,y,z)dy = -\iint_\sigma \frac{\partial Q}{\partial z}dydz + \iint_\sigma \frac{\partial Q}{\partial x}dxdy \qquad (9.70)$$

$$\oint_L R(x,y,z)dz = -\iint_\sigma \frac{\partial R}{\partial x}dzdx + \iint_\sigma \frac{\partial R}{\partial y}dydz \qquad (9.71)$$

tengliklarni yozish mumkin. (9.69), (9.70) va (9.71) tengliklarni hadma-had qo'shib (9.67) Stoks formulasini hosil qilamiz.◄

9.21-Misol. $\oint_L x^2y^3dx + dy + zdz$ integralni Stoks formulasi yordamida hisoblang, bu yerda L egri chiziq $x^2 + y^2 = 1$, $z = 0$ tenglamalar bilan aniqlangan aylana, σ sirt $x^2 + y^2 + y^2 = 1$ ($z > 0$) yarim sferaning yuqori qismidan iborat.

► $P(x,y,z) = x^2y^3$, $Q(x,y,z) = 1$ va $R(x,y,z) = z$ bo'lganligi uchun

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-3x^2y^2,\qquad \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0,\qquad \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=0$$

bo‘ladi. σ sirtning Oxy tekislikdagi proyeksiyasi $D_{xy}=\{(x,y): x^2+y^2\leq 1\}$ doiradan iborat. U holda Stoks formulasiga ko‘ra

$$\oint_L x^2y^3dx+dy+zdz=-3\iint_\sigma x^2y^2dxdy=-3\iint_{D_{xy}} x^2y^2dxdy$$

bu yerda $D_{xy}^*=\{(x,y): x^2+y^2\leq 1, x\geq 0, y\geq 0\}$ deb doiraning chorak qismini belgilaymiz va so‘ngi integral ostidagi funksiyaning ko‘rinishidan, hamda integrallash sohasining koordinata boshi va o‘qlariga nisbatan simmetrikligini inobatga olsak

$$-3\iint_{D_{xy}} x^2y^2dxdy=-12\iint_{D_{xy}^*} x^2y^2dxdy$$

bo‘ladi. Bu yerda qutb koordinatalariga o‘tamiz:

$$-12\iint_{D_{xy}^*} x^2y^2dxdy=-12\int_0^{\pi/2} d\varphi\int_0^1 r^2\cos^2\varphi\, r^2\sin^2\varphi\, rdr=$$

$$=-12\int_0^{\pi/2} \cos^2\varphi\sin^2\varphi\, d\varphi\int_0^1 r^5dr=$$

$$=-3\int_0^{\pi/2} \sin^2 2\varphi\, d\varphi\cdot\left.\frac{r^6}{6}\right|_0^1=-\frac{\pi}{8}.\ \blacktriangleleft$$

Nazorat savollari

1. $x=x(t), y=y(t), \alpha\leq t\leq\beta$ parametrik tenglama bilan berilgan L egri chiziq qanday shart bajarilganda silliq egri chiziq deb ataladi?
2. I tur egri chiziqli integralga ta’rif bering.
3. Qanday shart bajarilganda $f(x,y)$ funksiyaning L egri chiziq bo‘yicha I tur egri chiziqli integrali mavjud bo‘ladi?
4. I tur egri chiziqli integralning xossalarini sanab o‘ting.
5. L egri chiziq $x=x(t), y=y(t), \alpha\leq t\leq\beta$ parametrik tenglama bilan berilgan bo‘lsa, $f(x,y)$ funksiyaning L egri chiziq bo‘yicha I tur egri chiziqli integralini hisoblash formulasini keltiring.
6. L egri chiziq $y=y(x), a\leq x\leq b$ tenglama bilan berilgan bo‘lsa, $f(x,y)$ funksiyaning L egri chiziq bo‘yicha I tur egri chiziqli integralini hisoblash formulasini keltiring.
7. II tur egri chiziqli integralga ta’rif bering.

8. II tur egri chiziqli integral mavjudligi haqidagi teoremani bayon qiling.
9. II tur egri chiziqli integralning xossalarini bayon qiling.
10. Ostrogradskiy-Grin formulasini yozing.
11. II tur egri chiziqli integralning integrallash yo‘liga bog‘liq bo‘lmaslik shartini keltiring.
12. Qanday sirtlarga silliq sirt deb ataladi?
9. I tur sirt integraliga ta’rif bering.
14. Qanday shart bajarilganda I tur sirt integrali mavjud bo‘ladi?
15. I tur sirt integralini xossalarini bayon qiling.
16. Qanday sirtlarga oriyentirlangan sirtlar deb ataladi?
17. Bir tomonli sirtga misol keltiring.
18. II tur sirt integralining ta’rifini bayon qiling.
19. II tur sirt integralining xossalarini sanab o‘ting.
20. II tur sirt integralini hisoblash usulini keltiring.
21. Ostrogradskiy-Gauss formulasini yozing.
22. Stoks formulasini yozing.

Mashqlar.

Quyidagi I tur egri chiziqli integrallarni hisoblang:

1. $\int\limits_L (2x+y)ds$, bu yerda $L-$ chiziq $y=5x$ to‘g‘ri chiziqning $O(0,0)$ va $A(2,10)$ nuqtalar orasidagi kesmasi.

2. $\int\limits_L \frac{1}{x^2+y^2}ds$, bu yerda $L-$ chiziq $x^2+y^2=1$ aylananing yuqori yarmi.

3. $\int\limits_L \sqrt{x^3y}ds$, bu yerda $L-$chiziq $y=x^3$ giperbolaning $O(0,0)$ va $A(1,1)$ nuqtalarni tutashtiruvchi yoyi.

4. $y=x^2/2+1$ egri chiziqning $P(x,y)$ nuqtadagi zichligi $\mu(P)=x+1, x\in[0,\sqrt{3}]$ tenglik bilan aniqlansa, uning m massasini toping.

5. $x=10\cos^3 t\,, y=10\sin^3 t\,, 0\le t\le \pi/2$ astroidaning zichligi $\mu=1$ o‘zgarmas bo‘lsa, uning massasini va og‘irlik markazining koordinatalarini toping.

Quyidagi II tur egri chiziqli integrallarni hisoblang:

6. $\int\limits_{AO} x^3dx+3y^2zdy-x^2ydz$, bu yerda AB uchlari $A(3,2,1)$ va $O(0,0,0)$ nuqtalarda bo‘lgan kesma.

7. $\int\limits_{AB} 6x^2ydx + 10xy^2dy$, bu yerda integrallsh $y = x^3$ giperbolaning $A(1,1)$ nuqtadan $B(2,8)$ nuqtagacha bo‘lgan yoyi.

8. $\int\limits_{L} yzdx + xzdy + xydz$, bu yerda L egri chiziq $x = a\cos t$, $y = a\sin t$, $z = ct$ vint chizig‘ining t parametr 0 dan 2π gacha o‘zgargandagi qismi.

9. $\oint\limits_{L} -x^2ydx + xy^2dy$, bu yerda $L-$ markazi koordinatalar boshida va radiusi R bo‘lgan aylana.

10. $\int\limits_{(0,0)}^{1,1)} (x^2 + y)dx + (y^2 + x)dy$ egri chiziqli integral integrallash yo‘liga bog‘liq emasligiga ishonch hosil qilingandan so‘ng, uni hisoblang.

Quyidagi I tur sirt integrallarini hisoblang:

11. $\iint\limits_{\sigma} \frac{ds}{(1+x+z)^2}$, bu yerda $\sigma-$ sirt $x + y + z = 1$ tekislikning birinchi oktantda joylashgan qismi.

12. $\iint\limits_{\sigma} xyzds$, bu yerda $\sigma -$sirt $x + y + z = 1$ tekislikning birinchi oktantda joylashgan qismi.

9. $\iint\limits_{\sigma} (x + 18y + 24z)ds$, bu yerda $\sigma -$sirt $x + 2y + 3z = 1$ tekislikning birinchi oktantda joylashgan qismi.

14. $\iint\limits_{\sigma} xds$, bu yerda $\sigma -$ sirt $x^2 + y^2 = 1$ silindrning $z = 0$ va $z = 2$ tekisliklar bilan ajratilgan qismi.

15. $\iint\limits_{\sigma} z^4ds$, bu yerda $\sigma -$sirt $x^2 + y^2 + z^2 = 1$ sferaning yuqori yarim qismi.

Quyidagi II rur sirt integrallarini hisoblang:

16. $\iint\limits_{\sigma} x^2dydz + y^2dxdz + z^2dxdy$, bu yerda $\sigma -$ sirt $x^2 + y^2 + z^2 = 1$ $(y \geq 0)$ yarim sferaning tashqi tomoni.

17. $\iint\limits_{\sigma} xdydz + y^2dxdz + z^2dxdy$, bu yerda $\sigma -$ sirt $x^2 + y^2 \leq R^2, z \in [-H, H]$ jism sirtining tashqi tomoni.

18. $\iint\limits_{\sigma} xdydz + ydxdz + xdxdy$, bu yerda $\sigma -$sirt $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ellipsodning ichki tomoni.

19. $\iint\limits_{\sigma} (xy^2 + z^2)dydz + (yz^2 + x^2)dxdz + (zx^2 + y^2)dxdy$, bu yerda $\sigma -$sirt $x^2 + y^2 + z^2 = R^2$ $(z \geq 0)$ yarim sferaning tashqi tomoni.

20. $x^2+y^2=R^2, |z| \leq H$ silindrning yon sirtining tashqi tomoniga $\boldsymbol{v}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ tezlik vektoriga ega bo‘lgan suyuqlik oqimining vaqt birligida oqib o‘tgan suyuqlik hajmini toping (tekislik normali Oz o‘q bialn o‘tkir burchak tashkil qiladi).

Javoblar.

1. $2\sqrt{26}$. **2.** 2π. **3.** $(10\sqrt{10}-1)/54$. **4.** $m=7/3+\ln\sqrt{2+\sqrt{3}}+\sqrt{3}$. **5.** $m=5$, $C(4,4)$. **6.** $-87/4$. **7.** 3132. **8.** 0. **9.** $\pi R^4/2$. **10.** $5/3$. **11.** $\sqrt{3}(\ln 2-1/2)$. **12.** $\sqrt{3}/120$. **13.** $\sqrt{14}/2$. **14.** 0. **15.** $2\pi/5$. **16.** $\pi R^4/4$. **17.** $2\pi HR^2$. **18.** $-4\pi abc$. **19.** $\pi a^4(8a+5)/20$. **20.** $2\pi HR^2$

X BOB. MAYDONLAR NAZARIYASI ELEMENTLARI

Fizika, mexanika va matematikaning katta bo‘limi bo‘lgan maydonlar nazariyasida skalyar, vektor va tenzor maydonlar o‘rganiladi.

Fizika, elektrotexnika, mexanika, matematika va boshqa texnik fanlarning ko‘pgina masalalari skalyar va vektor maydonlarni o‘rganishga olib keladi.

10.1. Skalyar maydon tushunchasi

Agar fazoning yoki fazo qismining har bir nuqtasida biror miqdorning qiymatlari aniqlangan bo‘lsa, mazkur miqdorning *maydoni* berilgan deyiladi. Qaralayotgan miqdor skalyar bo‘lsa, ya’ni soniy qiymat bilan aniqlansa, maydon *skalyar maydon* deb ataladi. Misol sifatida qizdirilgan jismning ichki qismidagi temperaturalar maydonini, biror idishdagi suyuqlik bosimi maydonini, biror hajmdagi gaz zichligi maydonini va shunga o‘xshashlarni keltirish mumkin.

Skalyar maydon nuqtaning $u = u(M)$ funksiyasi bilan beriladi. Agar fazoda dekart koordinatalar sistemasi kiritilgan bo‘lsa, bu funksiya M nuqtaning x, y, z koordinatalarining funksiyasiga aylanadi:

$$u = u(x, y, z). \qquad (10.1)$$

10.1-Misol. $u = \sqrt{1 - x^2 - y^2 - z^2}$ funksiya radiusi 1 va markazi koordinatalar boshida bo‘lgan sfera bilan chegaralangan nuqtalarda skalyar maydonni aniqlaydi.

10.1-Ta’rif. $u(M)$ funksiya o‘zgarmas C qiymatni qabul qiladigan nuqtalar to‘plamiga maydonning *sath sirti* deb ataladi. Sath sirtining tenglamasi

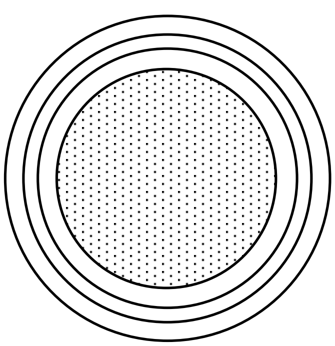
$$u(x, y, z) = C \qquad (10.2)$$

ko‘rinishda bo‘ladi.

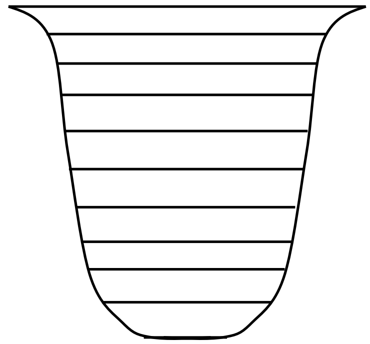
C o‘zgarmasga mumkin bo‘lgan barcha qiymatlarni berib, sath sirtlari oilasini hosil qilamiz.

10.2-Misol. Yer atmosferasi zichligi maydonining sath sirtlari markazi yer markazida bo‘lgan konsentrik sferalardan iborat bo‘ladi. Sfera radiusi qancha katta bo‘lsa, C o‘zgarmasning qiymati shuncha kichik bo‘lishi ravshan (10.1-rasm).

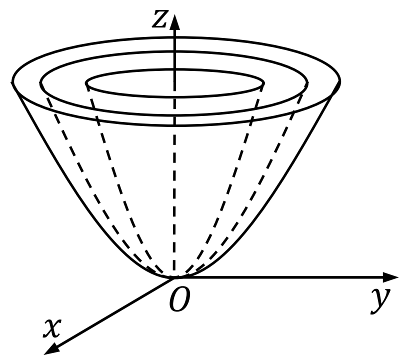
10.3-Misol. Idishdagi suvning gidrostatik bosimi maydonining sath sirtlari gorizontal tekisliklardan iborat bo‘ladi. Tekislikning chuqurligi qancha katta bo‘lsa, C o‘zgarmasning qiymati shuncha katta bo‘ladi (10.2-rasm).



10.1-rasm



10.2-rasm



10.3-rasm

10.4-Misol. $u = z/(x^2 + y^2)$ maydonni qaraymiz. Uning sath sirtlari

$$\frac{z}{x^2 + y^2} = C$$

ya'ni

$$z = C(x^2 + y^2)$$

tenglamaga ega bo‘ladi. Ular umumiy Oz aylanish o‘qiga ega bo‘lgan paraboloidlar oilasidan iborat. C o‘zgarmasning qiymati qancha katta bo‘lsa, paraboloid shuncha “tik” bo‘adi (10.3-rasm).

Skalyar maydonni aniqlovchi $u = u(M)$ funksiya Oxy tekislikda yoki uning bir qismida aniqlangan bo‘lsa, yassi skalyar maydonni hosil qilamiz. Uni M nuqtaning koordinatalari orqali ifodalasak, $u = u(x, y)$ funksiyaga ega bo‘lamiz. $u(x, y) = C$ bu holda maydon *sath chizig‘ining* tenglamasi deb ataladi.

10.2. Yo‘nalish bo‘yicha hosila

$u = u(x, y, z)$ skalyar funksiya bilan aniqlangan skalyar maydon berilgan bo‘lsin. Qo‘zg‘almas $M_0(x_0, y_0, z_0)$ va o‘zgaruvchan $M(x, y, z)$ nuqtalarni o‘lamiz (10.4-rasm). u funksiyaning bu nuqtalardagi qiymatlaridan

$$\Delta u(M_0) = u(M) - u(M_0),$$

ya'ni

$$\Delta u(x_0, y_0, z_0) = u(x, y, z) - u(x_0, y_0, z_0)$$

ayirmani tuzamiz.

$\vec{l} = \overrightarrow{M_0M}$ vektorning uzunligini Δl orqali belgilaymiz: $\Delta l = \left|\overrightarrow{M_0M}\right|$.

$$\frac{\Delta u}{\Delta l} = \frac{u(x, y, z) - u(x_0, y_0, z_0)}{\Delta l}$$

nisbat maydonning M_0M kesmada o‘zgarishining o‘rtacha tezligini aniqlaydi.

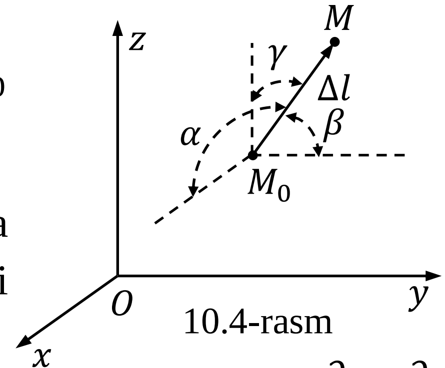
10.2-Ta’rif. Agar chekli

$$\lim_{\Delta l\to 0}\frac{\Delta u}{\Delta l}=\lim_{\Delta l\to 0}\frac{u(x,y,z)-u(x_0,y_0,z_0)}{\Delta l}$$

limit mavjud bo‘lsa, bu limitni biz $u(M)$ maydonning M_0 nuqtadagi $\vec{l}$ yo‘nalish bo‘yicha hosilasi deb ataymiz va uni $\frac{\partial u(M_0)}{\partial l}$ orqali belgilaymiz.

Fizik nuqtay nazardan bu miqdor M nuqta yo‘nalishida maydon o‘zgarishining M_0 nuqtadagi tezligini aniqlaydi.

$\Delta x=x-x_0,\ \Delta y=y-y_0,\ \Delta z=z-z_0$ deb belgilash kiritamiz. U holda $\Delta x=\Delta l\cos\alpha,\ \ \Delta y=\Delta l\cos\beta,\ \ \Delta z=\Delta l\cos\gamma$ bo‘ladi, bu yerda $\cos\alpha$, $\cos\beta$, $\cos\gamma$ yuqorida kiritilgan $\overrightarrow{M_0M}$ vektorning yo‘naltiruvchi kosinuslari.



10.4-rasm

$u(x,y,z)$ funksiya maydonning barcha nuqtalarida uzluksiz $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ va $\frac{\partial u}{\partial z}$ xususiy hosilalarga ega deb faraz qilamiz. U holda bu funksiya maydonda differensiallanuvchi bo‘ladi, bu esa

$$\Delta u(M_0)=\frac{\partial u(M_0)}{\partial x}\Delta x+\frac{\partial u(M_0)}{\partial y}\Delta y+\frac{\partial u(M_0)}{\partial z}\Delta z+o(\Delta l)$$

degan ma’noni anglatadi. Bu tenglikning ikkala tomonini Δl ga bo‘lib,

$$\frac{\Delta u(M_0)}{\Delta l}=\frac{\partial u(M_0)}{\partial x}\frac{\Delta x}{\Delta l}+\frac{\partial u(M_0)}{\partial y}\frac{\Delta y}{\Delta l}+\frac{\partial u(M_0)}{\partial z}\frac{\Delta z}{\Delta l}+\frac{o(\Delta l)}{\Delta l}$$

tenglikni hosil qilamiz. Yuqoridagi $\Delta x=\Delta l\cos\alpha$, $\Delta y=\Delta l\cos\beta$, $\Delta z=\Delta l\cos\gamma$ tengliklarni inobatga olsak, so‘ngi tenglik

$$\frac{\Delta u(M_0)}{\Delta l}=\frac{\partial u(M_0)}{\partial x}\cos\alpha+\frac{\partial u(M_0)}{\partial y}\cos\beta+$$
$$+\frac{\partial u(M_0)}{\partial z}\cos\gamma+\frac{o(\Delta l)}{\Delta l}\qquad(10.3)$$

ko‘rinishni oladi. $\lim_{\Delta l\to 0}\frac{o(\Delta l)}{\Delta l}=0$ ekanligini hisobga olib, (10.3) tenglikda $\Delta l\to 0$ bo‘lganda limitga o‘tamiz va natijada yo‘nalish bo‘yicha hosila uchun

$$\frac{\partial u(M_0)}{\partial l}=\frac{\partial u(M_0)}{\partial x}\cos\alpha+\frac{\partial u(M_0)}{\partial y}\cos\beta+\frac{\partial u(M_0)}{\partial z}\cos\gamma\qquad(10.4)$$

formulani hosil qilamiz.

Shunday qilib, biz $\dfrac{\partial u(M_0)}{\partial l}$ hosilaning nafaqat mavjudligini isbotladik, balki uni hisoblash formulasini ham keltirib chiqardik.

$\vec{l}$ yo‘nalish bo‘yicha hosila M_0 nuqtada maydonning bu yonalish bo‘yicha o‘zgarish tezligini tavsiflaydi. Agar $\dfrac{\partial u}{\partial l} > 0$ bo‘lsa, u funksiya $\vec{l}$ yo‘nalishda o‘sadi, agar $\dfrac{\partial u}{\partial l} < 0$ bo‘sa, u funksiya $\vec{l}$ yo‘nalishda kamayadi.

10.5-Misol. $u = x^2y + y^2z + z^2x$ maydonning $M_0(2,3,1)$ nuqtada $\overrightarrow{M_0M_1}$ vektor yo‘nalishidagi hosilasini toping, bu yerda $M_1(4,5,2)$.

►u funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = 2xy + z^2, \qquad \frac{\partial u}{\partial y} = 2yz + x^2, \qquad \frac{\partial u}{\partial x} = 2zx + y^2,$$

demak

$$\frac{\partial u(M_0)}{\partial x} = 13, \qquad \frac{\partial u(M_0)}{\partial y} = 10, \qquad \frac{\partial u(M_0)}{\partial z} = 13.$$

M_0 va M_1 nuqtalarning koordinatalaridan $\overrightarrow{M_0M_1} = \{2,2,1\}$ vektorni hosil qilamiz. Shuning uchun $\Delta l = \left|\overrightarrow{M_0M_1}\right| = \sqrt{2^2 + 2^2 + 1^2} = 3$ bo‘ladi. U holda $\overrightarrow{M_0M_1}$ vektorning yo‘naltiruvchi kosinuslari

$$\cos\alpha = \frac{2}{3}, \qquad \cos\beta = \frac{2}{3}, \qquad \cos\gamma = \frac{1}{3}$$

qiymatlarni qabul qiladi. Topilganlarni (10.4) formulaga qo‘yib

$$\frac{\partial u(M_0)}{\partial l} = 13 \cdot \frac{2}{3} + 10 \cdot \frac{2}{3} + 13 \cdot \frac{1}{3} = \frac{65}{3}$$

yo‘nalish bo‘yicha hosilaning qiymatini topdik.◄

Mulohaza. Agar $\alpha = 0, \beta = \gamma = 90^\circ$ bo‘lsa, (10.4) formula

$$\frac{\partial u(M_0)}{\partial l} = \frac{\partial u(M_0)}{\partial x}$$

ko‘rinishni oladi, ya’ni x bo‘yicha xususiy hosilani $u(M)$ maydondan Ox o‘qning musbat yo‘nalishi bo‘yicha olingan hosila sifatida qarash mumkin ekan. Xuddi shu singari mulohaza y va z bo‘yicha xususiy hosilalarga nisbatan ham o‘rinli bo‘ladi.

10.3. Skalyar maydonning gradiyenti va uning xossalari

Maydonning bitta nuqtadagi biroq turli yo‘nalishlardagi hosilalari, umuman olganda turlicha bo‘ladi. M_0 nuqtadagi $\frac{\partial u(M_0)}{\partial l}$ hosila qaysi yo‘nalishda (ya’ni qanday α, β va γ burchaklarda) eng katta qiymatni qabul qilishini aniqlaymiz.

(10.4) tenglikning o‘ng tomoni birlik $\vec{e} = \{\cos\alpha, \cos\beta, \cos\gamma\}$ vektor va boshqa $\vec{g} = \left\{\frac{\partial u(M_0)}{\partial x}, \frac{\partial u(M_0)}{\partial y}, \frac{\partial u(M_0)}{\partial z}\right\}$ vektorning skalyar ko‘paytmasidan iborat ekanligini ko‘rish mumkin:

$$\frac{\partial u(M_0)}{\partial l} = \vec{g}\vec{e}.$$

$|\vec{e}| = 1$ bo‘lganligi uchun

$$\frac{\partial u(M_0)}{\partial l} = |\vec{g}| \cos\left(\widehat{\vec{g}, \vec{e}}\right) \tag{10.5}$$

tenglikni yozish mumkin.

(10.5) tenglikdan $\frac{\partial u(M_0)}{\partial l}$ hosila $\vec{g}$ va $\vec{e}$ vektorlar orasidagi burchak nolga teng bo‘lganda eng katta qiymatni qabul qiladi degan xulosaga kelish mumkin. Bu esa qidirilayotgan yo‘nalish $\vec{g}$ vektorning yo‘nalishidan iborat ekanligidan dalolat beradi. Shunday qilib, $u(x, y, z)$ maydon M_0 nuqtada $\vec{g} = \left\{\frac{\partial u(M_0)}{\partial x}, \frac{\partial u(M_0)}{\partial y}, \frac{\partial u(M_0)}{\partial z}\right\}$ vektor yo‘nalishida eng tez o‘sar ekan. Bu vektor $u(M)$ skalyar maydonning M_0 nuqtadagi *gradiyenti* deb ataladi va grad $u(M_0)$ orqali belgilanadi.

Shunday qilib, ta’rifga ko‘ra

$$\text{grad}\, u(M_0) = \frac{\partial u(M_0)}{\partial x}\vec{i} + \frac{\partial u(M_0)}{\partial y}\vec{j} + \frac{\partial u(M_0)}{\partial z}\vec{k}. \tag{10.6}$$

Endi (10.4) tenglikni

$$\frac{\partial u(M_0)}{\partial l} = \vec{e} \cdot \text{grad}\, u(M_0)$$

yoki

$$\frac{\partial u(M_0)}{\partial l} = |\text{grad}\, u(M_0)| \cdot \cos\varphi \tag{10.7}$$

ko‘rinishda yozish mumkin, bu yerda φ kiritilgan grad $u(M_0)$ vektor va $\vec{l}$ yo‘nalish orasidagi burchak.

$u(x,y,z)$ maydonning $u(x,y,z)=C$ sath sirtiga M_0 nuqtada o‘tkazilgan urinma tekislik

$$\frac{\partial u(x_0,y_0,z_0)}{\partial x}(x-x_0)+\frac{\partial u(x_0,y_0,z_0)}{\partial y}(y-y_0)+$$
$$+\frac{\partial u(x_0,y_0,z_0)}{\partial z}(z-z_0)=0$$

tenglamaga ega. Demak $\frac{\partial u(M_0)}{\partial x},\frac{\partial u(M_0)}{\partial y},\frac{\partial u(M_0)}{\partial z}$ miqdoralar normal vektorining koordinatalari ekan. Shuning uchun (10.6) tenglik maydonning gradiyenti maydon sath sirtining berilgan nuqtadagi normali bo‘ylab yonalgan degan ma’noni anglatadi.

Skalyar maydon gradiyentining muhim xossalarini keltiramiz.

1. Gradiyent sath sirtining berilgan nuqtasidan o‘tuvchi normal bo‘ylab yo‘nalgan;
2. grad $(u+v)=$ grad $u+$ grad v;
3. grad $(c\cdot u)=c\cdot$ grad u, $c=$ const;
4. grad $(u\cdot v)=u\cdot$ grad $v+v\cdot$ grad u;
5. $grad\left(\frac{u}{v}\right)=\frac{v\cdot \text{grad}\, u-u\cdot \text{grad}\, u}{v^2}$;
6. grad $f(u)=\frac{\partial f}{\partial u}\cdot$ grad u.

►Bu xossalarning isboti gradiyentning ta’rifidan bevosita kelib chiqadi. Masalan, oxirgi xossani isbotlaylik. Ta’rifga ko‘ra

$$\text{grad}\, f(u)=\frac{\partial}{\partial x}(f(u))\vec{i}+\frac{\partial}{\partial y}(f(u))\vec{j}+\frac{\partial}{\partial z}(f(u))\vec{k}=$$
$$=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}\vec{i}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}\vec{j}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\vec{k}=\frac{\partial f}{\partial u}\cdot\text{grad}\, u \quad ◄$$

Mulohaza. Mydon grediyentining keltirilgan xossalari yassi maydon uchun ham o‘rinli.

10.6-Misol. $u=xy+yz+zx$ maydonning $A(1,-3{,}2)$ nuqtada o‘sishining eng yuqori tezligini toping.

►Dastlab maydonning gradiyentini topamiz:

$$\text{grad}\, u=(y+z)\vec{i}+(x+z)\vec{j}+(y+x)\vec{k};$$

$$\operatorname{grad} u(1,-3,2) = -\vec{i} + 3\vec{j} - 2\vec{k}.$$

Shuning uchun maydon o‘sishining eng katta tezligi

$$\operatorname{grad} u(A) = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{14}$$

bo‘ladi. ◄

10.4. Vektor maydon va vektor chiziqlar

Vektor maydonlarga ko‘rsatmali misol sifatida suyuqlik oqqanda tezlik maydonini ko‘rsatish mumkin. Fazoning biror D sohasida suyuqlik zarralarining harakati yuz bersin, bunda har bir $M \in D$ nuqta orqali vaqtning turli momentlarida o‘tgan suyuqlik zarralari bir xil $\vec{v}(M)$ tezlikka ega deb faraz qilamiz. Bu holda *suyuqlik oqimi statsionar* holatda deyiladi. Shunday qilib, statsionar holatdagi oqimda ixtiyoriy $M \in D$ nuqtada vaqt o‘tishi bilan $\vec{v}(M)$ vektor o‘zgarmaydi, ammo u turli M_1 va M_2 nuqtalarda $\vec{v}(M_1)$ va $\vec{v}(M_2)$ vektorlar turlicha bo‘lishi mumkin. Ana shu yo‘sinda D sohada vektor maydon−suyuqlik tezligi maydoni aniqlandi.

Agar fazoda $Oxyz$ dekart koordinatalar sistemasi berilgan bo‘lsa, vektor maydon uch o‘zgaruvchili vektor funksiya sifatida aniqlanishi mumkin. Bu holda D sohaning har bir $M(x,y,z)$ nuqtasida $\vec{a}(M)$ vektor maydonni

$$\vec{a}(M) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} \qquad (10.8)$$

tenglik bilan aniqlaymiz, bu yerda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ berilgan D sohada aniqlangan skalyar funksiyalar. By funksiyalarning $M(x,y,z)$ nuqtadagi qiymatlari $\vec{a}(M)$ vektorning koordinatalari bo‘ladi.

Agar vektor maydon biror to‘g‘ri burchakli $Oxyz$ koordinatalar sistemasida z o‘zgaruvchiga bog‘liq bo‘lmasa, uni biz *ikki o‘lchovli vektor maydon* deb ataymiz. Bir o‘lchovli vektor maydon esa ikki o‘lchovli vektor maydonning xususiy holi sifatida aniqlanadi. Ikki o‘lchovli bo‘lmagan maydonlarni *uch o‘lchovli vektor maydon* deb ataymiz. Har bir $\vec{a}(M)$ vektor Oxy tekislikka parallel bo‘lsa, u *yassi vektor maydon* deb ataladi, bu holda $R(x,y,z) \equiv 0$ bo‘ladi.

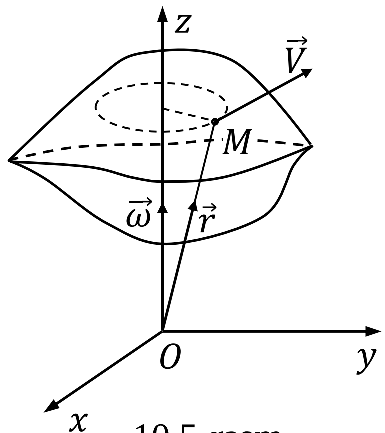
10.7-Misol. Aylanayotgan qattiq jism nuqtalarining tezliklari maydonini qaraymiz. Oz o‘qni aylanish o‘qi bo‘ylab yo‘naltitamiz (10.5-rasm). $\vec{\omega} = \{0,0,\omega\}$ vektor burchak tezlik bo‘lsin. U holda mexanikadagi ma’lum $\vec{v} = \vec{\omega} \times \vec{r}$ Eyler formulasidan

$$\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & \omega\\ x & y & z\end{vmatrix}=-\omega y\vec{i}+\omega x\vec{j}=\omega(-y\vec{i}+x\vec{j})$$

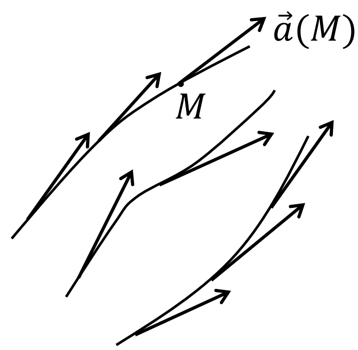
tenglikni hosil qilamiz.◀

(10.8) tenglik bilan aniqlanuvchi $\vec{a}(x,y,z)$ vektor maydon berilgan va uning $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ koordinatalari biror D sohada uzluksiz bo‘lsin.

Har bir nuqtasida o‘tkazilgan urinma $\vec{a}$ vektorga kollinear bo‘ladigan nuqtalarning geometrik o‘rniga bu maydonning *vektor chiziqlari* deb ataymiz (10.6-rasm). Elektr yoki magnit maydonlarida ular kuch chiziqlari deb yuritildadi.



10.5-rasm



10.6-rasm

$$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$$

vektor chiziqlardan birining radius vektori bo‘lsin. Vektor chiziqning ta’rifiga ko‘ra $\vec{a}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$ vektor va vektor chiziqqa o‘tazilgan

$$\frac{d\vec{r}}{dt}=\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}$$

urinma vektor ana shu chiziqning barcha nuqtalarida kollinear. U holda vektorlarning kollinearlik shartiga ko‘ra

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)} \tag{10.9}$$

tengliklar o‘rinli bo‘ladi. Shunday qilib, vektor chiziqlar uchun birinchi tartibli differensial tenglamalar sistemasini hosil qildik. Uni integrallab $\vec{a}$ maydonning ikki parametrli vektor chiziqlar oilasini hosil qilamiz.

10.8-Misol. $\vec{a}=x\vec{i}+y\vec{j}+2z\vec{k}$ vektor maydonning vektor chiziqlarini toping.

►Vektor chiziqlarning differensial tenglamalar sistemasini tuzamiz:

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2z}$$

yoki

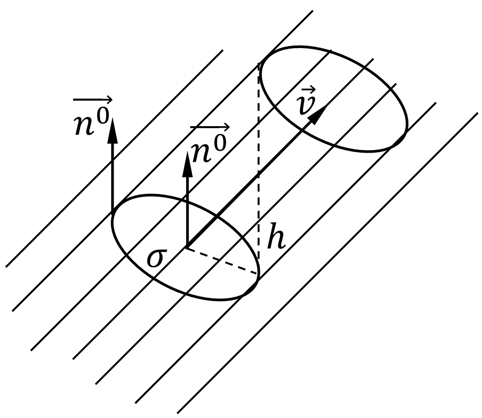
$$\frac{dx}{x}=\frac{dy}{y}, \qquad \frac{dx}{x}=\frac{dz}{2z}.$$

Bu sistemani integrallab, ikkita $y=c_1x,\ z=c_2x^2$ tenglamani hosil qilamiz, bu yerda c_1, c_2 ixtiyoriy o‘zgarmas sonlar.

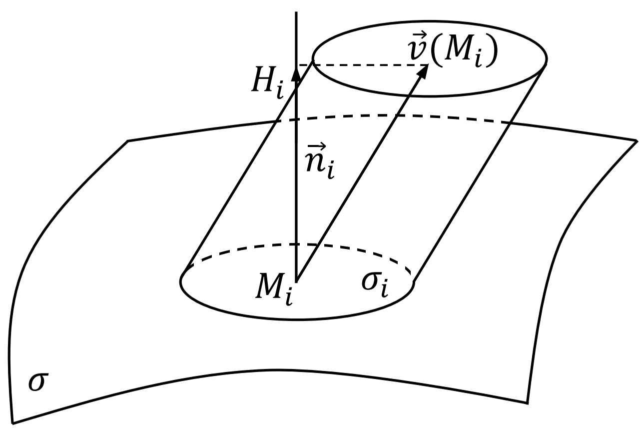
$y=c_1x$ tekislik bilan $z=c_2x^2$ parabolik silindrning kesishuvi ikki parametrli vektor chiziqlar oilasini beradi.◄

10.5. Vektor maydon oqimi va divergensiyasi

Dastlab xususiy hol bo‘lgan suyuqlik oqimidagi $\vec{v}$ tezliklar maydonini qaraymiz. Maydonda biror σ sirtni ajratamiz. Vaqt birligi davomida σ sirt orqali oqib o‘tuvchi suyuqlik miqdoriga *suyuqlikning σ sirt orqali oqimi* deyiladi.



10.7-rasm



10.8-rasm

Agar oqimning tezligi o‘zgarmas ($\vec{v}=\text{const}$), σ sirt esa yassi bo‘lsa, bu oqimni hisoblash oson. Bu holda suyuqlik oqimi parallel asosli va yasovchisining uzunligi $|\vec{v}|$ bo‘lgan silindrik jismning hajmiga teng, chunki vaqt birligi davomida har bir zarra $\vec{v}$ miqdorga suriladi (10.7-rasm):

$$O=Sh,$$

bu yerda $S-$ asos yuzi, $h=Pr_{\vec{n}}\vec{v}=\vec{v}\overrightarrow{n^0}-$ silindrning balandligi va $\vec{n}-$silindr asosining normali, $\left|\overrightarrow{n^0}\right|=1$.

Shunday qilib, o‘zgarmas $\vec{v}$ tezlikda yassi σ sirt orqali o‘tuvchi suyuqlik oqimi

$$O = \vec{v}\overrightarrow{n^0} \cdot S \qquad (10.10)$$
miqdorga teng bo‘ladi.

Agar $\vec{v}$ tezlik uzluksiz o‘zgarsa, σ sirt esa silliq bo‘lsa, sirtni shunday kichik σ_i $(i = 1,2,\dots,n)$ qismlarga ajratish mumkinki, bunda har bir σ_i qismni yassi va undagi $\vec{v}$ vektorni o‘zgarmas deb qarash mumkin (10.8-rasm). U holda vaqt birligi davomida σ_i qism orqali taqriban $O_i \approx H_i \cdot \Delta s_i$ miqdorda suyuqlik oqib o‘tadi, $\Delta s_i -$ bu yerda σ_i qismning yuzi, H_i yasovchisi $\vec{v}(M_i)$ bo‘lgan $i -$ silindrning balandligi, M_i esa σ_i qismning biror nuqtasi. H_i balandlikni $\vec{v}(M_i)$ vektorning $\vec{n}(M_i)$ normaldagi proyeksiyasi deb qarash mumkin: $H_i = Pr_{\vec{n}(M_i)}\vec{v}(M_i) = \vec{v}(M_i) \cdot \vec{n}_i$, bu yerda $\vec{n}_i$ sirtga M_i nuqtada o‘tkazilgan normalning birlik vektori. σ sirt orqali oqib o‘tuvchi suyuqlik miqdori uning σ_i qismlari orqali o‘tuvchi suyuqlik miqdorlari yig‘indisiga teng bo‘lganligi uchun
$$O \approx \sum_{i=1}^{n} \vec{v}(M_i) \cdot \vec{n}_i \cdot \Delta s_i \qquad (10.11)$$
taqribiy tenglikni yozish mumkin.

Suyuqlikning qidirilayotgan aniq miqdorini topilgan yig‘indining σ_i qismlar sonini cheksiz orttirilgandagi va ularning o‘lchamlarini (σ_i qismning d_i diametrini) nolga intiltirilgandagi limiti sifatida hosil qilish mumkin:
$$O = \lim_{\substack{n\to\infty \\ (\max d_i \to 0)}} \sum_{i=1}^{n} \vec{v}(M_i) \cdot \vec{n}_i \cdot \Delta s_i = \iint_{\sigma} \vec{v}(M) \cdot \vec{n} ds \qquad (10.12)$$

10.3-Ta’rif. $\vec{a}$ vektorning σ sirt orqali O_σ oqimi deb, vektor maydon va sirt normali birlik vektorlari sklayar ko‘paytmasining sirt bo‘yicha integraliga aytiladi, ya’ni
$$O_\sigma = \iint_{\sigma} \vec{a}\vec{n} ds. \qquad (10.13)$$

Agar $\vec{n} = \{\cos\alpha ; \cos\beta ; \cos\gamma\}$ va $\vec{a} = \{P; Q; R\}$ bo‘lsa, $\vec{a}$ vektor maydonning σ sirt orqali oqimini
$$O_\sigma = \iint_{\sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma) ds \qquad (10.14)$$
ko‘rinishda ifodalash mumkin.

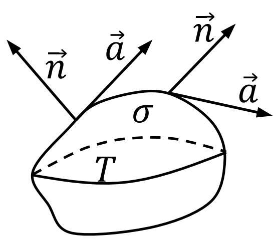
I va II tur sirt integrallari orasidagi bog‘liqlikni ifodalovchi (13.58) formuladan foydalanib vektor oqimini
$$O_\sigma = \iint_{\sigma} Pdydz + Qdxdz + Rdxdy \qquad (10.15)$$
II tur sirt integrali orqali ifodalaymiz.

Vektor oqim skalyar miqdor ekanligini ta'kidlab o'tamiz.

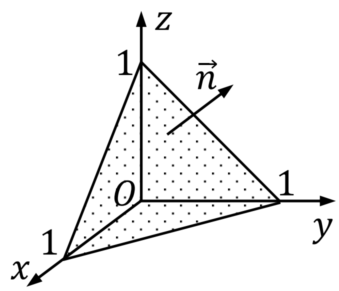
σ sirt yopiq va biror T sohani chegaralagan holni alohida qarab chiqamiz. Bu holda $\vec{a} = \{P; Q; R\}$ vektor maydonning σ sirt orqali oqimini

$$O_\sigma = \oiint_\sigma Pdydz + Qdxdz + Rdxdy \tag{10.16}$$

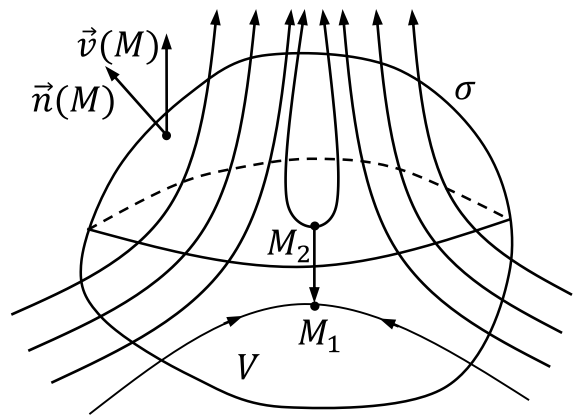
orqali belgilaymiz. Sirt yopiq bo'lganda odatda $\vec{n}$ vektorning yo'nalishi sifatida tashqi normalning yo'nalishi olinadi (10.9-rasm).



10.9-rasm



10.10-rasm



10.11-rasm

10.9-Misol. $\vec{a} = x\vec{i} + y\vec{j}$ vektor maydonning $x + y + z = 1$ tekislikning birinchi oktantdagi qismi (normal Oz o'q bilan o'tkir burchak tashkil qiladi) orqali oqimini toping (10.10-rasm).

►$F(x, y, z) = 0$ tenglama bilan berilgan sirtning birlik normallari

$$\vec{n} = \pm \frac{\text{grad } F}{|\text{grad } F|}$$

formula bilan topiladi. Shuning uchun berilgan tekislikning birlik normali

$$\vec{n} = \pm \frac{1}{\sqrt{3}} \left(\vec{i} + \vec{j} + \vec{k}\right)$$

vektordan iborat bo'ladi. Normal Oz o'q bilan o'tkir burchak tashkil qilishi lozimligini inobatga olsak, ya'ni $\cos\gamma = \pm 1/\sqrt{3} > 0$ ekanligidan, musbat ishorani olamiz. Demak

$$\vec{n} = \frac{1}{\sqrt{3}} \left(\vec{i} + \vec{j} + \vec{k}\right).$$

U holda $\vec{a}\vec{n}$ skalyar ko'paytma uchun

$$\vec{a}\vec{n} = \frac{1}{\sqrt{3}} (x + y)$$

tenglikni hosil qilamiz. Qidirilayotgan vektor oqimni (10.13) formulaga ko‘ra hisoblaymiz:

$$O_\sigma = \iint_\sigma \frac{1}{\sqrt{3}}(x+y)ds.$$

σ sirtning Oxy tekislikdagi proyeksiyasi $D = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 1-x\}$ uchburchakdan iborat bo‘ladi. Shuning uchun sirt integralidan ikki o‘lchovli integralga o‘tib, oqimni hisoblaymiz:

$$O_\sigma = \iint_D \frac{1}{\sqrt{3}}(x+y)\frac{dxdy}{|\cos\gamma|} = \iint_D (x+y)dxdy = \int_0^1 dx \int_0^{1-x} (x+y)dy =$$

$$= \int_0^1 \left[\left(xy + \frac{1}{2}y^2\right)\Big|_0^{1-x}\right]dx = \int_0^1 \left(x(1-x) + \frac{1}{2}(1-x)^2\right)dx =$$

$$= \frac{1}{2}\int_0^1 (1-x^2)dx = \frac{1}{2}\left(x - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2}\left(1-\frac{1}{3}\right) = \frac{1}{3}.$$

Demak, $O_\sigma = 1/3$ oqim birligi.◄

Vektor maydonni suyuqlik oqimi tezliklari maydoni sifatida talqin qilamiz. Bu holda vektor maydonning σ yopiq sirt bo‘yicha oqimining musbat qiymati σ chegaralagan sohaga oqib kirgan suyuqlik miqdoriga nisbatan oqib chiqqan suyuqlik miqdorining ko‘pligini anglatadi. Demak, sohada suyuqlik hosil bo‘ladigan (masalan,qor yoki muz eriganda suv hosil bo‘ladi) nuqta yoki qism sohalar mavjud. Bunday nuqtalarni *vektor maydon manbalari* deb ataymiz. Xuddi shu singari oqimning manfiy qiymati sohadan oqib chiqish sohaga oqib kirishga nisbatan kamligini anglatadi. Demak, sohada suyuqlik yo‘qoladigan (masalan, bug‘lanadi yoki muzlaydi) nuqtalar mavjud. Bunday nuqtalar *vektor maydonning singish nuqtalari* deb ataladi.

Manba va singishlar nuqtaviy yoki taqsimlangan bo‘lishi mumkin. Sohada nuqtaviy manbalarning mavjudligi sohada boshlanadigan vektor chiziqlarning mavjudligidan, nuqtaviy singishlarning mavjudligi esa sohada tugaydigan vektor chiziqlrning mavjudligidan dalolat beradi. 10.11-rasmda vektor chiziqlar boshlanadigan $M_2 -$ manba, vektor chiziqlar tugaydigan $M_1 -$singish nuqtasi bo‘ladi.

Vaqt birligida paydo bo‘ladigan yoki yo‘qoladigan suyuqlik hajmiga manbaning yoki singishning *intensivligi* deb ataymiz. Singish manfiy intensivli manba deb qarab, singishlarning intensivligini manfiy qiymatli deb hisoblash qulay. U holda tezliklar vektor maydonining σ sirt orqali O oqimini bu sirt chegaralagan sohadagi barcha manba va singishlar intensivliklarining yig‘indisi deb talqin qilish mumkin.

Turli fizik masalalrda manba (singish) va uning intensivligi turlicha ma’noni anglatadi. Masalan, elektr maydon kuchlanishi vektor maydonida manba (singish) musbat (manfiy) zaryadlardan, ularning intensivligi esa bu zaryadlar miqdoridan iborat bo‘ladi. Agar vektor maydon issiqlik oqimini ifodalasa, manba va singishlar issiqlik ajralishini va yutilishini, manbaning (singishning) intensivligi esa vaqt birligida ajraladigan (yutiladigan) issiqlik miqdorini anglatadi.

Bektor maydon manbalari taqsimlangan holni qarab chiqamiz. Bunday manbalar fazoning biror sohasi, biror sirti yoki biror chiziq bo‘ylab taqsimlangan bo‘lishi mumkin. Agar manbalar biror yopiq σ sirt bilan chegaralangan D soha bo‘ylab taqsimlangan bo‘lsa, vektor maydonning σ sirt orqali O_σ oqimining D soha V hajmiga nisbati vektor maydonning D sohadagi manbalarining o‘rtacha zichligini beradi. D sohada qo‘zg‘almas M nuqtani olamiz va $d_D \to 0$ (d_D $-$bu yerda D sohaning diametri) bo‘lganda limitga o‘tamiz:

$$\lim_{d_D \to 0} \frac{O_\sigma}{V} = \lim_{d_D \to 0} \frac{1}{V} \iint_\sigma \vec{a}\vec{n}ds. \qquad (10.17)$$

Agar bu limit mavjud bo‘lsa, uning qiymati vektor maydon taqsimlangan manbasining M nuqtadagi intensivligini beradi.

(10.17) limit mavjud bo‘lsa, uni biz $\vec{a}$ vektor maydonning $M \in D$ nuqtadagi *divergensiyasi* (lotincha *divergentia*-farqlanish) deb ataymiz va div $\vec{a}(M)$ orqali belgilaymiz.

$\vec{a}$ vektor maydonning $M \in D$ nuqtadagi divergensiyasi skalyar miqdor. $\vec{a}$ vektor maydonning aniqlanish sohasidagi barcha nuqtalardagi divergensiyasi div $\vec{a}$ skalyar maydonni aniqlaydi.

Vektor maydonning berilgan nuqtadagi divergensiyasi vektor maydon oqimining hajmga nisbatining limiti sifatida aniqlanadi va u koordinatalar sistemasining tanlanishiga bog‘liq emas. Biroq divergensiyani hisoblashda birorta koordinatalar sistemasidan foydalanishga to‘g‘ri keladi. Biz to‘g‘ri burchakli koordinatalar sistemasida vektor maydonning divergensiyasini hisoblash masalasini qarab chiqamiz.

$\vec{a}(M)$ vektor maydon fazoviy D sohada differensiallanuvchi, ya’ni biror $Oxyz$ to‘g‘ri burchakli koordinatalar sistemasida differensiallanuvchi

$$\vec{a}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + P(x,y,z)\vec{k}$$

vektor funksiya bilan aniqlangan bo‘lsin. D sohada yopiq σ sirtni olamiz va sirtning M nuqtasida tashqi normalning birlik vektorini $\vec{n} = \{\cos\alpha\,;\cos\beta\,;\cos\gamma\}$ yo‘naltiruvchi kosinuslar yordamida aniqlaymiz.

U holda (10.17) tenglikdagi integralda $\vec{a}\vec{n}$ skalyar ko‘paytmani koordinatalar orqali ifodalab va unga (13.63) Ostrogradskiy-Gauss formulasini qo‘llasak

$$\oiint_{\sigma} \vec{a}\vec{n}ds = \oiint_{\sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds =$$
$$= \iiint_{D} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dV \qquad (10.18)$$

formulani hosil qilamiz.

$\dfrac{\partial P}{\partial x}, \dfrac{\partial Q}{\partial y}, \dfrac{\partial R}{\partial z}$ xususiy hosilalarning D sohada uzluksizligini inobatga olgan holda uch o‘lchovli integral uchun o‘rta qiymat haqidagi teoremani qo‘llab, vektor maydonning ixtiyoriy M nuqtadagi divergensiyasi uchun

$$\operatorname{div}\vec{a}(M) = \lim_{d_D\to 0}\frac{1}{V}\iiint_{D}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dV =$$
$$= \frac{\partial P(M)}{\partial x} + \frac{\partial Q(M)}{\partial y} + \frac{\partial R(M)}{\partial z} \qquad (10.19)$$

tenglikka ega bo‘lamiz. Shunday qilib, to‘g‘ri burchakli koordinatalar sistemasida divergensiyani

$$\operatorname{div}\vec{a}(x,y,z) = \frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z} \qquad (10.20)$$

formula bilan hisoblash mumkin ekan.

Divergensiyaning ba’zi xossalarini keltiramiz.

1. Agar $\vec{a}$ $-$o‘zgarmas vektor bo‘lsa, $\operatorname{div}\vec{a} = 0$ bo‘ladi.

2. $\operatorname{div}(c\cdot\vec{a}) = c\cdot\operatorname{div}\vec{a}$, bu yerda $c = \text{const}$.

3. $\operatorname{div}\left(\vec{a} + \vec{b}\right) = \operatorname{div}\vec{a} + \operatorname{div}\vec{b}$, ya’ni ikkita vektor maydon yig‘indisining divergensiyasi, bu vektor maydonlar divergensiyalarining yig‘indisiga teng.

4. Agar U $-$skalyar mydon, $\vec{a}$ $-$vektor maydon bo‘lsa,

$$\operatorname{div}(U\cdot\vec{a}) = U\cdot\operatorname{div}\vec{a} + \vec{a}\operatorname{grad}U$$

bo‘ladi.

►Bu xossalarni (10.20) formula yordamida oson isbotlash mumkin. Masalan, 4-xossani isbotlaymiz.

$U\cdot\vec{a}=U\cdot P\vec{i}+U\cdot Q\vec{j}+U\cdot R\vec{k}$ tenglikka (10.20) formulani qo‘llaymiz:

$$\operatorname{div}(U\cdot\vec{a})=\frac{\partial}{\partial x}(U\cdot P)+\frac{\partial}{\partial y}(U\cdot Q)+\frac{\partial}{\partial z}(U\cdot R)=$$

$$=U\frac{\partial P}{\partial x}+P\frac{\partial U}{\partial x}+U\frac{\partial Q}{\partial y}+Q\frac{\partial U}{\partial y}+U\frac{\partial R}{\partial z}+R\frac{\partial U}{\partial z}=$$

$$=U\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)+P\frac{\partial U}{\partial x}+Q\frac{\partial U}{\partial y}+R\frac{\partial U}{\partial z}=$$

$$=U\cdot\operatorname{div}\vec{a}+\vec{a}\operatorname{grad}U.$$ ◄

10.10-Misol. $\vec{\omega}=\{0,0,\omega\}$ burchak tezlik bilan Oz o‘q atrofida aylanayotgan qattiq jism nuqtalari $\vec{v}$ tezlik maydonining divergensiyasini hisoblang.

►10.8-Misolda bu vektor maydon

$$\vec{v}=-\omega y\vec{i}+\omega x\vec{j}=\omega(-y\vec{i}+x\vec{j})$$

vektor funksiya bilan aniqlanishini ko‘rgan edik. U holda (10.20) formulaga ko‘ra

$$\operatorname{div}\vec{v}=\frac{\partial(-\omega y)}{\partial x}+\frac{\partial(\omega x)}{\partial y}=0.$$

Shunday qilib, aylanayotgan jismning ixtiyoriy nuqtasidagi tezliklar maydonining divergensiyasi nolga teng ekan.◄

Vektor maydonning oqimi va divergensiyasi tushunchalari Ostrogradskiy-Gauss formulasini qisqa vektor shaklda ifodalash imkonini beradi. Buning uchun formulada sirt integralini qandaydir vektor maydonning oqimi sifatida, uch o‘lchovli integralni esa xuddi shu vektor maydon divergensiyasining integrali deb qarash kerak.

10.1-Teorema (Ostrogradskiy-Gauss). Agar $\vec{a}(M)$ vektor maydon D fazoviy sohada differensiallanuvchi bo‘lsa, u holda σ yopiq bo‘lakli silliq sirt bilan chegaralangan ixtiyoriy $V\subset D$ soha uchun

$$\oiint\limits_{\sigma}\vec{a}\vec{n}ds=\iiint\limits_{V}\operatorname{div}\vec{a}dV \qquad (10.21)$$

formula o‘rinli bo‘ladi, $\vec{n}$ bu yerda σ $-$sirtning tashqi birlik normali.

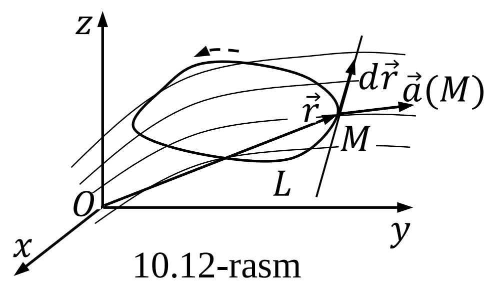
Fizik nuqtay nazardan Ostrogradskiy-Gauss teoremasi vektor maydonning yopiq sirt bo‘ylab oqimi sirt chegaralagan sohadagi barcha manba va singishlar intensivliklari yig‘indisiga tengligini anglatadi.

10.6. Vektor maydon sirkulyatsiyasi va rotori

Fazoviy D sohada

$$\vec{a}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} \qquad (10.22)$$

vektor maydon aniqlangan va bu sohada yopiq L silliq chiziq berilgan bo‘lsin. Bu chiziqda soat strelkasi harakatiga qarama-qarshi yo‘nalishni musbat yo‘nalish deb tanlaymiz.



10.12-rasm

L konturdagi M nuqtaning radius vektori $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ bo‘lsin. Ma’lumki

$$d\vec{r} = dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k}$$

vektor L chiziqqa o‘tkazilgan urinmada chiziqda tanlangan yo‘nalish bo‘ylab yo‘nalgan bo‘ladi (10.12-rasm). U holda $|d\vec{r}| = dl$ bo‘ladi, bu yerda dl −chiziq yoyining differensiali: $dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

10.4-Ta’rif. $\vec{a}$ vektor maydon va $d\vec{r}$ urinma vektorlar skalyar ko‘paytmasining L kontur bo‘yicha olingan egri chiziqli integrali $\vec{a}$ vektor maydonning L kontur bo‘ylab *sirkulyatsiyasi* (lotincha *sirkulyatsiya*-aylanish) deb ataladi va $C_{\vec{a}}$ orqali belgilanadi. Demak

$$C_{\vec{a}} = \oint_L \vec{a}\, d\vec{r}. \qquad (10.23)$$

Integral ostidagi $\vec{a}\, d\vec{r}$ skalyar ko‘paytmani ularning koordinatalari orqali ifodalasak, sirkulyatsiya uchun

$$C_{\vec{a}} = \oint_L Pdx + Qdy + Rdz \qquad (10.24)$$

formulani hosil qilamiz. (10.24) formula oddiy fizik ma’noga ega: agar L kontur kuch maydonida yotsa, sirkulyatsiya material nuqta L chiziq bo‘ylab kochganda $\vec{a}(M)$ kuch maydoni bajargan ishidan iborat.

10.11-Misol. $\vec{a}(x,y,z) = -2z\vec{i} - x^2\vec{j} + z\vec{k}$ vektor maydonning

$$\begin{cases} x = 2\cos t, \\ y = 2\sin t, \quad t \in [0,2\pi] \\ z = \sin t, \end{cases}$$

parametrik tenglama bilan berilgan L kontur bo‘ylab sirkulyatsiyasini hisoblang.

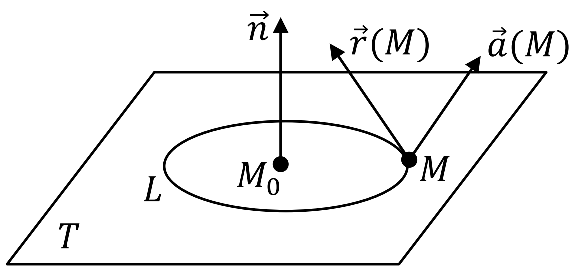
►Sirkulyatsiyani (10.24) formuladagi integralni aniq integralga o‘tkazib hisoblaymiz:

$$C_{\vec{a}} = \oint_L (-2z)dx - x^2dy + zdz =$$

$$= \int_0^{2\pi} [-2\sin t \cdot (-2\sin t) - 4\cos^2 t \cdot 2\cos t + \sin t \cdot \cos t]dt =$$

$$= 4\int_0^{2\pi} \sin^2 t\, dt - 8\int_0^{2\pi} \cos^3 t\, dt + \int_0^{2\pi} \sin t dt = 4\pi. \blacktriangleleft$$

Kuch maydoni sirkulyatsiyasining nol qiymatni qabul qilishi maydon material nuqta ko‘chganda boshlang‘ich holatiga qaytariladigan ish bajarganligini anglatadi. Bu holda kuch maydoni *uyurmali* xususiyatga ega deb aytiladi. Kuch maydoni qaerda va qanday darajada uyurmali xusiyatga ega ekanligini aniqlaymiz. Birorta $\vec{n}$ birlik vektor, $\vec{a}$ vektor maydonning aniqlanish sohasida M_0 nuqtani tanlaymiz. $\vec{n}$ vektorga perpendikulyar va M_0 nuqtadan o‘tuvchi T tekislikda M_0 nuqtani o‘rab turuvchi bo‘lakli silliq



10.13-rasm

L chiziqni olamiz. Bu L chiziq chegaralab turgan sohaning yuzini F orqali belgilaymiz (10.13-rasm). Kontur bo‘yicha sirkulyatsiyaning bu kontur chegaralagan soha yuziga nisbatining

$$\omega_{\vec{n}}(M_0) = \lim_{L \to M_0} \frac{1}{F} \oint_L \vec{a}\, d\vec{r} \qquad (10.25)$$

limiti mavjud bo‘lsa, uni biz vektor mydonning M_0 nuqtada $\vec{n}$ vektor yo‘nalishidagi *uyurmalanganlik darajasi* deb ataymiz.

$\vec{a}(M)$ vektor maydonning M_0 nuqtada $\vec{n}$ birlik vektor yo‘nalishidagi uyrmalanganlik darajasini birorta vektorning $\vec{n}$ birlik vektor yo‘nalishidagi proyeksiyasi sifatida aniqlash mumkin. Bu vektor $\vec{a}(M)$ vektor maydonning M_0 nuqtadagi rotori (ba’zan uyurmasi) deb ataladi (lotincha *roto*- aylantiraman) va $\mathrm{rot}\, \vec{a}(M)$ orqali belgilanadi. Proyeksiyaning ta’rifidan uyurmalanganlik darajasi $\vec{n}$ birlik vektor o‘rniga $\mathrm{rot}\, \vec{a}(M)$ vektor olinsa eng katta qiymatni qabul qiladi. Shunday qilib, vektor maydonning M_0 nuqtadagi rotori vektor maydon uyurmalanganlik darajasi bu nuqtada eng katta bo‘ladigan yo‘nalishni aniqlaydigan vektor ekan.

Vektor maydon sirkulyatsiyasi va rotori koordinatalar sistemasining tanlanishiga bog‘liq emas. Biroq ularni hisoblash vektor maydon birorta koordinatalar sistemasida berilgandagina mumkin bo‘ladi. $Oxyz$ koordinatalar sistemasida

$$\vec{a}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

vektor funksiya bilan aniqlanuvchi $\vec{a}(M)$ vektor maydon rotori

$$\operatorname{rot}\vec{a}(M)=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{\imath}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{\jmath}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k} \qquad (10.26)$$

formula bilan hisoblanadi. Bu formulani eslab qolish uchun qulay bo‘lgan

$$\operatorname{rot}\vec{a}(M)=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

simvolik ko‘rinishda ham yozish mumkin.

10.12-Misol. $\vec{a}(x,y,z)=zy^2\vec{\imath}+xz^2\vec{\jmath}+yx^2\vec{k}$ vektor maydonning rotorini toping.

►Bu yerda $P=zy^2, Q=xz^2, R=yx^2$, u holda (10.26) formulaga ko‘ra

$$\operatorname{rot}\vec{a}(M)=(x^2-2xz)\vec{\imath}+(y^2-2xy)\vec{\jmath}+(z^2-2yz)\vec{k}. \blacktriangleleft$$

(10.26) formula yordamida rotorning quyidagi xossalarini isbotlash mumkin.

1. Agar $\vec{a}$ −o‘zarmas vektor bo‘lsa, $\operatorname{rot}\vec{a}=0$ bo‘ladi.

2. O‘zgarmas ko‘paytuvchini rotor belgisidan tashqariga chiqarish mumkin:

$$\operatorname{rot}(c\cdot\vec{a})=c\cdot\operatorname{rot}\vec{a}.$$

3. Ikkita vektor maydon yig‘indisining rotori bu vektor maydonlar rotorlarining yig‘indisiga teng:

$$\operatorname{rot}\left(\vec{a}+\vec{b}\right)=\operatorname{rot}\vec{a}+\operatorname{rot}\vec{b}.$$

4. Agar U −skalyar funksiya, $\vec{a}(M)$ esa vektor maydon bo‘lsa,

$$\operatorname{rot}(U\cdot\vec{a})=U\cdot\operatorname{rot}\vec{a}+\operatorname{grad}U\times\vec{a}$$

tenglik o‘rinli bo‘ladi.

Egri chiziqli integral bilan sirt integralini bog‘lovchi (13.67)

$$\oint_L Pdx+Qdy+Rdz=$$

$$=\iint_\sigma\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx$$

Stoks formulasini vektor maydonning sirkulyatsiyasi va rotori tushunchalari yordamida vektor shaklda yozamiz. Bu formulaning chap tomoni $\vec{a}(M)$ vektor maydonning L kontur bo‘ylab $C_{\vec{a}}$ sirkulyatsiyasidan iborat, ya’ni

$$\oint_L Pdx+Qdy+Rdz=\oint_L\vec{a}\,d\vec{r}.$$

O‘ng tomondagi integral esa rot $\vec{a}$ vektor maydonning σ sirt orqali O oqimidan iborat, ya’ni

$$\iint_{\sigma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx$$
$$= \iint_{\sigma} (\mathrm{rot}\vec{a})\vec{n}ds.$$

Demak Stoks formulasini

$$\oint_L \vec{a}\, d\vec{r} = \iint_{\sigma} (\mathrm{rot}\vec{a})\vec{n}ds \qquad (10.27)$$

vektor ko‘rinishda ifodalash mumkin. Bu yerda Stoks teoremasidagi singari L konturdagi musbat yo‘nalish bilan σ sirtdagi tomonning tanlanishi o‘zaro muvofiqlashtirilgan.

Shunday qilib, quyidagi teorema o‘rinli.

10.2-Teorema (Stoks). $\vec{a}(M)$ vektor maydonning L kontur bo‘ylab sirkulyatsiyasi bu vektor maydon rotorining L kontur chegaralagan σ sirt orqali oqimiga teng.

10.7. Gamilton operatori

Birinchi tartibli vektor differensial operatorlar. $u(x, y, z)$ skalyar funksiyaning gradiyenti ko‘pincha ∇ (“nabla”) harfi bilan belgilanadi va

$$\nabla u = \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z} \qquad (10.28)$$

ko‘rinishda yoziladi. (10.28) tenglikni simvoli ravishda

$$\nabla u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) u \qquad (10.29)$$

shaklda yozish mumkin.

10.5-Ta’rif. Ushbu

$$\nabla u = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

simvol *Gamilton operatori* (yoki *nabla-operator* yoki *simvolik vektor*) deb ataladi.

(10.28) va (10.29) formulalardan skalyar funksiyaning gradiyenti ∇ simvolik vektor bilan u sklayar funksiyaning “ko‘paytmasidan” iborat ekanligi kelib chiqadi, ya’ni $\nabla u = \mathrm{grad}\, u$.

∇ simvolik vektor bilan $\vec{a} = P\vec{i} + Q\vec{j} + R\vec{k}$ vektorlarning skalyar va vektor ko‘paytmalarini topamiz. Birinchi holda

$$\nabla\vec{a} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(P\vec{i} + Q\vec{j} + R\vec{k}\right) =$$

$$= \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}\vec{a}.$$

Demak

$$\nabla\vec{a} = \operatorname{div}\vec{a} \qquad (10.30)$$

tenglikka ega bo‘lamiz, ya’ni $\vec{a}$ vektorning divergensiyasi ∇ simvolik vektorning $\vec{a}$ vektorga sklayar ko‘paytmasiga teng bo‘lar ekan.

Ikkinchi holda esa

$$\nabla\times\vec{a} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(P\vec{i} + Q\vec{j} + R\vec{k}\right) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \times\left(P\vec{i} + Q\vec{j} + R\vec{k}\right) =$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

Demak

$$\nabla\times\vec{a} = \operatorname{rot}\vec{a}, \qquad (10.31)$$

ya’ni $\vec{a}$ vektorning rotori ∇ simvolik vektor bilan $\vec{a}$ vektorning vektor ko‘paytmasiga teng bo‘lar ekan.

Ikkinchi tartibli vektor differensial operatorlar. $T \subset \boldsymbol{R}^3$ sohada $u(M)$ skalyar maydon va $\vec{a}(M)$ vektor maydonlar berilgan bo‘lsin. $\operatorname{grad} u$, $\operatorname{div}\vec{a}$, $\operatorname{rot}\vec{a}$ operatsiyalarni *birinchi tartibli operatsiyalar* deb ataymiz. Birinchi va uchinchi operatsiyalar vektor maydonni, ikkinchisi esa skalyar maydonni yuzaga keltiradi.

Bu hosil bo‘lgan maydonlar ustida yana $\operatorname{rot}\operatorname{grad} u$, $\operatorname{div}\operatorname{grad} u$, $\operatorname{grad}\operatorname{div}\vec{a}$, $\operatorname{div}\operatorname{rot}\vec{a}$, $\operatorname{rot}\operatorname{rot}\vec{a}$ operatsiyalarni bajarish mumkin va ular *ikkinchi tartibli operatsiyalar* deb ataladi. Bu ikkinchi tartibli operatsiyalar uchun quyidagi munosbatlar o‘rinli:

$$\operatorname{rot}\operatorname{grad} u = 0, \ \operatorname{div}\operatorname{rot}\vec{a} = 0. \qquad (10.32)$$

Haqiqatan ham, (10.6), (10.26) formulalarga va ikkinchi tartibli aralash hosilalarning tenglik xossasiga ko‘ra

$$\operatorname{rot}\operatorname{grad} u = \left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 u}{\partial x \partial z}\right)\vec{j} +$$
$$+\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}\right)\vec{k} = 0,$$

(10.20), (10.26) formulalarga va ikkinchi tartibli aralash hosilalarning tenglik xossasiga ko‘ra

$$\operatorname{div}\operatorname{rot}\vec{a} = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) =$$
$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0.$$

10.6-Ta’rif. Ikkinchi tartibli $\operatorname{div}\operatorname{grad} u$ operatsiyasi Laplas operatori deb ataladi va Δ orqali belgilanadi (orttirma belgisi bilan adashtirmang), ya’ni $\Delta u = \operatorname{div}\operatorname{grad} u$.

Endi Laplas operatori uchun ifodani keltirib chiqaramiz:

$$\operatorname{div}\operatorname{grad} u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Shunday qilib,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \qquad (10.32)$$

Qolgan ikkita $\operatorname{grad}\operatorname{div}\vec{a}$, $\operatorname{rot}\operatorname{rot}\vec{a}$ ikkinchi tartibli operatsiyalar uchun (10.6), (10.20) va (10.26) formulalardan

$$\operatorname{grad}\operatorname{div}\vec{a} =$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\vec{i} + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\vec{j} + \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\vec{k} =$$
$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x \partial y} + \frac{\partial^2 R}{\partial x \partial z}\right)\vec{i} + \left(\frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial y \partial z}\right)\vec{i}$$
$$+\left(\frac{\partial^2 P}{\partial z \partial x} + \frac{\partial^2 Q}{\partial z \partial y} + \frac{\partial^2 R}{\partial z^2}\right)\vec{k},$$

$$\operatorname{rot}\operatorname{rot}\vec{a} = \left(\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\right)\vec{i} +$$
$$+\left(\frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)\vec{j} +$$

$$+\left(\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\right)\vec{k}=$$

$$=\left(\frac{\partial^2 Q}{\partial y\partial x}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial z\partial x}\right)\vec{i}+\left(\frac{\partial^2 R}{\partial z\partial y}-\frac{\partial^2 Q}{\partial z^2}-\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 P}{\partial x\partial y}\right)\vec{j}+$$

$$=\left(\frac{\partial^2 P}{\partial x\partial z}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}+\frac{\partial^2 Q}{\partial y\partial z}\right)\vec{k}$$

tengliklarni hosil qilamiz.

Nazorat savollari

1. Skalyar maydonga ta’rif bering.
2. Skalyar maydonning sath sirti nima?
3. Yo‘nalish bo‘yicha hosilaga ta’rif bering.
4. Yo‘nalish bo‘yicha hosila bilan xususiy hosilaning farqi nimada?
5. Skalyar maydonning gradiyenti qanday xossalarga ega?
6. Vektor maydonga ta’rif bering?
7. Skalyar maydon va vektor maydonlarning farqi nimada?
8. Vektor maydonning oqimi qanday aniqlanadi?
9. Vektor maydon divergensiyasiga ta’rif bering.
10. Ostrogradskiy-Gauss formulasi nimalarning tengligini ifodalaydi?
11. Vektor maydon sirkulyatsiyasiga ta’rif bering.
12. Vektor maydon rotoriga ta’rif bering.
13.Stoks teoremasini bayon qiling.
10. Gamilton operatorining ko‘rinishini yozing.
15. Qanday ikkinchi tartibli vektor differensial operatorlar mavjud?

Mashqlar.

$u(x,y,z)$ skalyar maydon va $M_0(x_0,y_0,z_0), M_1(x_1,y_1,z_1)$ nuqtalar berilgan. Maydonning M_0 nuqtada $\overrightarrow{M_0M_1}$ vektor yo‘nalishidagi hosilasini toping.

1. $u=(x+y+z)x^2$, $M_0(-2,1,1), M_1(5,-3,4)$.
2. $u=x^2y^3z^4$, $M_0(1,-1,2), M_1(2,-2,4)$.
3. $u=x^2+y^2+z^2$, $M_0(0,-1,1), M_1(1,1,-2)$.
4. $u=xy^2z^3$, $M_0(2,-1,1), M_1(3,-1,2)$.

$u(x,y,z)$ skalyar maydonning $M_0(x_0,y_0,z_0)$ nuqtadagi gradiyentini hisoblang.

5. $u = 2xy + 3yz + 4x^2y^2z^2$, $M_0(5,1,-2)$.
6. $u = x^2yz + xy^2z + xyz^2$, $M_0(3,4,1)$.
7. $u = e^xyz$, $M_0(0,1,2)$. **8.** $u = xy\sin z$, $M_0(1,2,\pi/3)$.

$\vec{a} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ vektor maydonning vektor chiziqlarini toping.

9. $\vec{a} = 2y\vec{i} + 6x\vec{j}$. **10.** $\vec{a} = z\vec{i} - x\vec{k}$. **11.** $\vec{a} = 2y\vec{j} + 3z\vec{k}$. **12.** $\vec{a} = 3x\vec{i} + 6y\vec{j}$.

$\vec{a}$ vektor maydonning berilgan tekislikning birinchi oktantdagi qismi (normal Oz o‘q bilan o‘tkir burchak tashkil qiladi) orqali oqimini toping.

13. $\vec{a} = x\vec{i} + y\vec{j}$, $x + y + z = 1$. **14.** $\vec{a} = 2x\vec{i} + z\vec{k}$, $x + y + z = 1$.
15. $\vec{a} = y\vec{j} + z\vec{k}$, $2x + y + z = 1$. **16.** $\vec{a} = x\vec{i} + y\vec{j}$, $x + y + 2z = 1$.

$\vec{a}$ vektor maydonning $M_0(x_0, y_0, z_0)$ nuqtadagi divergensiyasini hisoblang.

17. $\vec{a} = xy^2\vec{i} + yz^2\vec{j} + zx^2\vec{k}$, $M_0(2,-1,2)$.
18. $\vec{a} = 5y^2\vec{i} + 4z^2\vec{j} + 2x^2\vec{k}$, $M_0(1,2,-1)$.
19. $\vec{a} = (x + y^2 + 3z)\vec{i} + (y - z^2)\vec{j} + (z - x^2)\vec{k}$, $M_0(1,-1,1)$.
20. $\vec{a} = (x + y + z)\vec{i} + (x^2 + y^2 + z^2)\vec{j} + (x^3 + y^3 + z^3)\vec{k}$, $M_0(3,-1,1)$.

$\vec{a}$ vektor maydonning berilgan L kontur bo‘ylab sirkulyatsiyasini hisoblang.

21. $\vec{a} = y\vec{i} - z^2\vec{j} + x^2y\vec{k}$, $L = \{x = 2\cos t, y = \sin t, z = 1, t \in [0,2\pi]\}$.
22. $\vec{a} = -x^2y\vec{i} + 4\vec{j} + x^2z\vec{k}$, $L = \{x = 2\cos t, y = 2\sin t, z = 4, t \in [0,2\pi]\}$.
23. $\vec{a} = x^3\vec{i} - z^2\vec{j} + y\vec{k}$, $L = \{x = \cos t, y = 2\sin t, z = 2\cos t, t \in [0,2\pi]\}$.
24. $\vec{a} = -2z\vec{i} - x^2\vec{j} + z\vec{k}$, $L = \{x = 2\cos t, y = 2\sin t, z = \sin t, t \in [0,2\pi]\}$.

$\vec{a}$ vektor maydonning rotorini toping.

25. $\vec{a} = 3xy^2\vec{i} + 2yz^2\vec{j} - 5zx^2\vec{k}$. **26.** $\vec{a} = (x - 2y)\vec{i} + (x + 2z)\vec{j} - 5zx^2\vec{k}$.
27. $\vec{a} = xyz\vec{i} + (x - yz^2)\vec{j} + z(x + y)\vec{k}$. **28.** $\vec{a} = xz^2\vec{i} + yx^2\vec{j} - zx^2\vec{k}$.

Javoblar.

1. $\frac{24}{\sqrt{74}}$. **2.** $24\sqrt{6}$. **3.** $-\frac{10}{\sqrt{14}}$. **4.** $\frac{7}{\sqrt{2}}$. **5.** $\text{grad}u(M_0) = 162\bar{\imath} + 804\bar{\jmath} - 397\bar{k}$. **6.** $\text{grad}u(M_0) = 44\bar{\imath} + 36\bar{\jmath} + 108\bar{k}$. **7.** $\text{grad}u(M_0) = 2\bar{\imath} + 2\bar{\jmath} + \bar{k}$. **8.** $\text{grad}u(M_0) = \sqrt{3}\bar{\imath} + \frac{\sqrt{3}}{2}\bar{\jmath} + \bar{k}$. **9.** $\begin{cases} 3x^2 - y^2 = C_1, \\ z = C_2. \end{cases}$ **10.** $\begin{cases} x^2 + z^2 = C_1, \\ y = C_2. \end{cases}$ **11.** $\begin{cases} y = C_1\sqrt[3]{z^2}, \\ x = C_2. \end{cases}$ **12.** $\begin{cases} y = C_1 x^2, \\ z = C_2. \end{cases}$ **13.** $1/6$ oqim birligi. **14.** $1/2$ oqim birligi. **15.** $1/6$ oqim birligi. **16.** $1/6$ oqim birligi. **17.** $\text{div}\bar{a}(M_0) = 9$. **18.** $\text{div}\bar{a}(M_0) = 0$. **19.** $\text{div}\bar{a}(M_0) = 3$. **20.** $\text{div}\bar{a}(M_0) = 2$. **21.** -2π. **22.** 4π. **23.** -4π . **24.** 4π . **25.** $\text{rot}\bar{a} = -4yz\bar{\imath} - 10xz\bar{\jmath} - 6xy\bar{k}$. **26.** $\text{rot}\bar{a} = -2\bar{\imath} - 10xz\bar{\jmath} - \bar{k}$. **27.** $\text{rot}\bar{a} = z(1-2y)\bar{\imath} + (xy - z)\bar{\jmath} + (1 - xz)\bar{k}$. **28.** $\text{rot}\bar{a} = 4xz\bar{\jmath} + 2xy\bar{k}$.

XI. BOB. QATORLAR NAZARIYASI

Cheksiz ko‘p sondagi qo‘shiluvchilarning yig‘indisini o‘rganish muhim amaliy ahamiyatga ega.

11.1. Sonli qator ta’rifi va uning yaqinlashishi

Haqiqiy yoki kompleks sonlarning

$$u_1, u_2, \dots, u_n, \dots \tag{11.1}$$

cheksiz ketma-ketligi berilgan bo‘lsin.

11.1-Ta’rif. (11.1) ketma-ketlik hadlaridan tuzilgan

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{11.2}$$

ifoda sonli qator deb ataladi. Bunda ketma-ketlikning $u_1, u_2, \dots, u_n, \dots$ elemantlari qatorning hadlari, ketma-ketlikning umumiy u_n elementi esa qatorning umumiy hadi deyiladi.

(11.2) qatorni qisqa qilib

$$\sum_{n=1}^{\infty} u_n$$

ko‘rinishda ham yozish mumkin.

11.1-Misol. Umumiy hadi $u_n = (2n-1)/(3n+4)$ formula bilan berilgan $\sum_{n=1}^{\infty} u_n$ sonli qatorning dasrlabki uchta hadini yozamiz.

►Birinchi u_1 hadni qatorning umumiy hadiga $n = 1$ qiymatni qo‘yib topamiz, ya’ni

$$u_1 = \left.\frac{2n-1}{3n+4}\right|_{n=1} = \frac{2\cdot 1 - 1}{3\cdot 1 + 4} = \frac{1}{7}.$$

Ikkinchi u_2 hadni umumiy hadga $n = 2$ qiymatni qo‘yib topamiz:

$$u_2 = \left.\frac{2n-1}{3n+4}\right|_{n=2} = \frac{2\cdot 2 - 1}{3\cdot 2 + 4} = \frac{3}{10}.$$

Xuddi shu singari

$$u_3 = \left.\frac{2n-1}{3n+4}\right|_{n=3} = \frac{2\cdot 3 - 1}{3\cdot 3 + 4} = \frac{5}{13}.$$

Shunday qilib berilgan qator

$$\sum_{n=1}^{\infty} u_n = \frac{1}{7} + \frac{3}{10} + \frac{5}{13} + \cdots$$

ko‘rinishda bo‘lar ekan.◄

11.2-Ta’rif. (11.2) qatorning dastlabki n ta hadining

$$s_n = u_1 + u_2 + \dots + u_n$$
yig‘indisini bu qatorning n −qismiy yig‘indisi deb ataymiz.

Birinchi, ikkinchi, uchinchi va hokazo

$$s_1 = u_1,$$
$$s_2 = u_1 + u_2,$$
$$s_3 = u_1 + u_2 + u_3,$$
$$\dots\dots\dots\dots\dots\dots\dots$$

qismiy yig‘indialar cheksiz ketma-ketlikni tashkil qiladi.

11.3-Ta’rif. Qismiy yig‘indilarining $s_1, s_2, \dots, s_n, \dots$ ketma-ketligi chekli

$$\lim_{n\to\infty} s_n = s$$

limitga intilsa, (11.2) qator yaqinlashuvchi va s bu qatorning yig‘indisi deyiladi.

11.4-Ta’rif. Agar (11.2) qatorning qismiy yig‘indilari ketma-ketligi chekli limitga intilmasa yoki cheksizlikga intilsa, bu qator uzoqlashuvchi deyiladi.

Qatorlar nazariyasining mazmuni u yoki bu qatorning yaqinlashish yoki uzoqlashishini aniqlash va yaqinlashuvchi qatorning yig‘indisini hisoblashdan iborat. Har bir qatorning yaqinlashuvchi yoki uzoqlashuvchi ekanligini isbotlashni, hamda qator yig‘indisini hisoblashni bevosita ta’rifdan kelib chiqqan holda bajarish mumkin. Buning uchun qator n −qismiy yig‘indisining analitik ifodasini tuzib, n o‘sganda bu ifodaning limitini topish kerak bo‘ladi.

11.2-Misol. Umumiy hadi $u_n = 1/3^n$ formula bilan beriladigan

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots + \frac{1}{3^n} + \dots$$

qatorning n −qismiy yigindisi uchun analitik ifoda

$$s_n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

ko‘rinishda bo‘ladi va demak

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{3^n}\right) = \frac{1}{2},$$

yoki

$$\sum_{i=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots + \frac{1}{3^n} + \dots = \frac{1}{2}.$$

ya’ni qator yaqinlashuvchi va uning yig‘indisi $1/2$ ga teng.◄

11.3-Misol. Ushbu

$$1+2+3+\cdots+n+\cdots$$

qatorning $n-$qismiy yig‘indisi uchun analitik ifoda

$$s_n=1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

ko‘rinishda bo‘ladi va

$$s_1=1, s_2=3, s_3=6, \ldots, s_n=\frac{n(n+1)}{2}, \ldots$$

qismiy yigindilar ketma-ketligi cheksiz o‘sadi, ya’ni qator uzoqlashuvchi.◄

11.4-Misol. Hadlari 1 va -1 qiymatdan iborat

$$1-1+1-1+\cdots$$

qatorning qismiy yig‘indilari n juft bo‘lganda 0 va toq bo‘lganda 1 qiymatga teng bo‘ladi:

$$s_1=1, s_2=0, s_3=1, s_4=0, \ldots$$

va bu ketma-ketlik chegaralangan bo‘lganiga qaramay, limitga ega emas. Shuning uchun berilgan qator uzoqlashuvchi bo‘ladi.◄

11.5-Misol. Hadlari geometrik prograessiyani tashkil qiluvchi

$$\sum_{n=1}^{\infty} q^{n-1}, q\in\mathbb{R}$$

geometrik qatorning yaqinlashishini tekshiramiz.

►Geometrik progressiyaning dastlabki n ta hadining yig‘indisi formulasidan

$$s_n=\sum_{k=1}^{n} q^{k-1}=1+q+q^2+\cdots+q^{n-1}=\begin{cases}\dfrac{1-q^n}{1-q}, q\neq 1;\\ \quad n, \qquad q=1\end{cases}$$

tenglikni hosil qilamiz. q parametrning turli qiymatlarida s_n qismiy yig‘indining limitini hisoblaymiz. Agar $q=1$ bo‘lsa,

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty} n=\infty,$$

ya’ni qator uzoqlashuvchi bo‘ladi. $q=-1$ bo‘lganda 11.4-Misolda qaralgan uzoqlashuvchi qator hosil bo‘ladi.

Endi $q\neq\pm1$ holni qaraymiz. $|q|<1$ bo‘lganda berilgan qatorning hadlari cheksiz kamayuvchi geometrik prograessiyani tashkil qiladi. Ma’lumki cheksiz kamayuvchi geometrik prograeesiyaning yigindisi uchun

$$1+q+q^2+\cdots+q^n+\cdots=\frac{1}{1-q}$$

tenglik o‘rinli. Nihoyat $|q| > 1$ bo‘lganda

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \lim_{n\to\infty} \left(\frac{1}{1-q} - \frac{q^n}{1-q}\right) = \infty,$$

ya’ni bu holda ham qator uzoqlashuvchi bo‘ladi. Shunday qilib berilgan qator $|q| \geq 1$ holda uzoqlashuvchi va $|q| < 1$ holda yig‘indisi

$$\sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q}$$

bo‘lgan yaqinlashuvchi qatorga ega bo‘lamiz.◄

11.6-Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

qator yig‘indisini toping.

►Qatorning umumiy hadini

$$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

ko‘rinishda yozib olamiz. U holda qatorning s_n qismiy yig‘indisi

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$= 1 - \frac{1}{n+1}$$

tenglikni qanoatlantiradi. Shuning uchun qator yig‘indisi uchun

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

tenglikni hosil qilamiz.◄

11.2. Qoldiq qator

Yuqorida qaralgan

$$u_1 + u_2 + \cdots + u_n + \cdots \qquad (11.2)$$

qator berilgan bo‘lsin.

11.5-Ta’rif. (11.2) qatorning dastlabki n ta hadini tashlab yuborishdan hosil bo‘lgan

$$u_{n+1} + u_{n+2} + \cdots \qquad (11.3)$$

qator (11.2) qatorning n −qoldiq qatori deb ataladi.

(11.3) qatorning yig‘indisini r_n orqali belgilaymiz:

$$r_n = u_{n+1} + u_{n+2} + \cdots$$

(11.3) qoldiq qatorning $m-$qismiy yig‘indisi dastlab berilgan (11.2) qator qismiy yig‘indilarining $s_{n+m}-s_n$ ayirmasiga teng. $s_{n+m}=s_n+(s_{n+m}-s_n)$ tenglikda m bo‘yicha limitga o‘tamiz:

$$\lim_{m\to\infty} s_{n+m}=s_n+\lim_{n\to\infty}(s_{n+m}-s_n). \tag{11.4}$$

Bu tenglikning chap tomonidagi limit (11.2) berilgan qatorning yig‘indisiga teng, o‘ng tomondagi limit esa r_n qoldiq qator yig‘indisiga teng. U holda chap tomondagi limitning mavjudligidan o‘ng tomondagi limitning mavjudligi kelib chiqadi va aksincha. Shuning uchun birorta qoldiq qator yaqinlashsa, berilgan qator ham yaqinlashadi. Xuddi shu singari qator yaqinlashishidan uning barcha qoldiq qatorlarining ham yaqinlashishi kelib chiqadi. Bundan esa, agar (11.2) qator yaqinlashuvchi bo‘lsa,

$$s=s_n+r_n \tag{11.5}$$

tenglik kelib chiqadi.

11.1-Teorema. Agar (11.2) qator yaqinlashsa, u holda uning r_n qoldiq qatori n o‘sishi bilan nolga intiladi.

►(11.5) tenglik ixtiyoriy n uchun o‘rinli, shuning uchun bu yerda limitga o‘tishimiz mumkin:

$$s=\lim_{n\to\infty}(s_n+r_n)=\lim_{n\to\infty}s_n+\lim_{n\to\infty}r_n.$$

Yaqinlashuvchi qator uchun $\lim\limits_{n\to\infty}s_n=s$ ekanligidan bu tenglikdan

$$\lim_{n\to\infty}r_n=0$$

kelib chiqadi.◄

11.3. Qator yaqinlashishining zaruriy sharti

Qator yaqnlashishing eng sodda zaruriy sharti quyidagi teoremada keltirilgan.

11.2-Teorema. Agar (11.2) qator yaqinlashsa, uning u_n umumiy hadi no‘lga intiladi: $\lim\limits_{n\to\infty}u_n=0$.

►(11.2) qatorning $(n-1)$ va $n-$qismiy yig‘indilari uchun $s_n=s_{n-1}+u_n$ tenglik o‘rinli. U holda yaqinlashuvchi qator uchun o‘rinli bo‘lgan $\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}s_{n-1}=s$ tengliklarni inobatga olgan holda bu tenglikda limitga o‘tsak

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}(s_{n-1}+u_n)=\lim_{n\to\infty}s_{n-1}+\lim_{n\to\infty}u_n$$

yoki

$$s = s + \lim_{n\to\infty} u_n$$

tenglik hosil bo‘ladi, bundan esa $\lim\limits_{n\to\infty} u_n = 0$ tenglik kelib chiqadi.◄

11.7-Misol. Ushbu

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$$

qator uzoqlashuvchi. Chunki uning uchun

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$$

bo‘ladi, ya’ni qator yaqinlashishining zaruriy sharti o‘rinli emas. Shuning uchun berilgan qator uzoqlashusvchi.◄

11.2-Teoremada keltirilgan shart qator yaqinlashishi uchun zaruriy, biroq yetarli emas, ya’ni $\lim\limits_{n\to\infty} u_n = 0$ tenglik o‘rinli bo‘lishiga qaramasdan berilgan qator uzoqlashuvchi bo‘lishi mumkin.

11.8-Misol. Ushbu

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

qatorning umumiy hadi nolga intiladi:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

va n −qismiy yig‘ndisi uchun

$$s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq n\frac{1}{\sqrt{n}} = \sqrt{n}, n \in \mathbb{N}$$

tengsizlik o‘rinli. Shuning uchun bu tengsizlikga ko‘ra $\lim\limits_{n\to\infty} \sqrt{n} = +\infty$ ekanligidan $\lim\limits_{n\to\infty} s_n = +\infty$ munosabat kelib chiqadi, ya’ni berilgan qator uzoqlashuvchi.◄

11.4. Yaqinlashuvchi qatorlarning xossalari

11.3-Teorema. Agar

$$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

qator yaqinlashuvchi va $\lambda \neq 0$ biror o‘zgarmas son bo‘lsa, u holda

$$\lambda u_1 + \lambda u_2 + \cdots + \lambda u_n + \cdots = \sum_{n=1}^{\infty} \lambda u_n$$

qator ham yaqinlashuvchi bo‘ladi va

$$\sum_{n=1}^{\infty} \lambda u_n = \lambda \sum_{n=1}^{\infty} u_n$$

tenglik o‘rinli bo‘ladi.

► $\sum_{n=1}^{\infty} u_n$ va $\sum_{n=1}^{\infty} \lambda u_n$ qatorlarning n −qismiy yig‘indilarini tuzamiz:

$$s_n = u_1 + u_2 + \cdots + u_n, \qquad \bar{s}_n = \lambda u_1 + \lambda u_2 + \cdots + \lambda u_n.$$

Bu qismiy yig‘indilar uchun $\bar{s}_n = \lambda s_n$ ekanligi ravshan. Teorema shartiga ko‘ra $\sum_{n=1}^{\infty} u_n$ qator yaqinlashuvchi, ya’ni $\lim_{n\to\infty} s_n$ limit mavjud. U holda so‘ngi tenglikga ko‘ra $\lim_{n\to\infty} \bar{s}_n$ limit ham mavjud bo‘ladi va

$$\lim_{n\to\infty} \bar{s}_n = \lim_{n\to\infty} \lambda s_n = \lambda \lim_{n\to\infty} s_n$$

yoki

$$\sum_{n=1}^{\infty} \lambda u_n = \lambda \sum_{n=1}^{\infty} u_n. ◄$$

Hadlari $\sum_{n=1}^{\infty} u_n$ va $\sum_{n=1}^{\infty} v_n$ qatorlarning bir xil tartib raqamli hadlarining yig‘indisidan iborat bo‘lgan

$$\sum_{n=1}^{\infty} (u_n + v_n) = (u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n) + \cdots$$

qatorga $\sum_{n=1}^{\infty} u_n$ va $\sum_{n=1}^{\infty} v_n$ qatorlarning yig‘indisi deb ataladi.

11.4-Teorema. Agar $\sum_{n=1}^{\infty} u_n$ va $\sum_{n=1}^{\infty} v_n$ qatorlar yaqinlashuvchi bo‘lsa, ularning $\sum_{n=1}^{\infty} (u_n + v_n)$ yig‘indisi ham yaqinlashuvchi bo‘ladi va

$$\sum_{n=1}^{\infty} (u_n + v_n) = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n \qquad (11.6)$$

tenglik o‘rinli bo‘ladi.

►Berilgan qatorlarning n −qismiy yig‘indilarini tuzamiz: $U_n = u_1 + u_2 + \cdots + u_n$, $V_n = v_1 + v_2 + \cdots + v_n$, u holda yig‘indi qatorning n −qismiy yig‘indisi

$$S_n = (u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n) = U_n + V_n$$

bo‘ladi. $S_n = U_n + V_n$ tenglikning o‘ng tomoni limitga ega bo‘lsa, chap tomoni ham limitga ega bo‘ladi va

$$\sum_{n=1}^{\infty} (u_n + v_n) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} U_n + \lim_{n\to\infty} V_n = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n$$

(11.6) tenglikga ega bo‘lamiz.◄

11.1-Natija. Agar $\sum_{n=1}^{\infty} u_n$ va $\sum_{n=1}^{\infty} v_n$ qatorlar yaqinlashuvchi bo‘lsa, ularning $\sum_{n=1}^{\infty} (\alpha u_n + \beta v_n)$ yig‘indisi ham yaqinlashuvchi bo‘ladi va

$$\sum_{n=1}^{\infty} (\alpha u_n + \beta v_n) = \alpha \sum_{n=1}^{\infty} u_n + \beta \sum_{n=1}^{\infty} v_n$$

tenglik o‘rinli bo‘ladi.

►11.3-Teoremaga ko‘ra $\sum_{n=1}^{\infty} \alpha u_n$ va $\sum_{n=1}^{\infty} \beta v_n$ qatorlar yaqinlashadi va 11.4-Teoremaga ko‘ra ularning yig‘indisi $\sum_{n=1}^{\infty} (\alpha u_n + \beta v_n)$ qator yig‘indisiga teng bo‘ladi.◄

11.5-Teorema (Koshi kriteriyasi). $\sum_{n=1}^{\infty} u_n$ qator yaqinlashishi uchun ixtiyoriy $\varepsilon > 0$ soni uchun shunday $N = N(\varepsilon)$ tartib raqami topilib, ixtiyoriy $n > N$ va $p = 0, 1, 2, \dots$ qiymatlarda

$$\left|u_n + u_{n+1} + \cdots + u_{n+p}\right| < \varepsilon \tag{11.7}$$

tengsizlikning bajarilishi zarur va yetarli.

►Qaralayotgan $\sum_{n=1}^{\infty} u_n$ qatorning s_{n+p} va s_{n-1} qismiy yig‘indilaridan foydalanib, (11.7) tengsizlikni

$$\left|s_{n+p} - s_{n-1}\right| < \varepsilon$$

ko‘rinishda yozish mumkin. Bu tengsizlikning bajarilishi $\{s_n\}$ qismiy yig‘indilar ketma-ketligi uchun Koshi kriteriyasining o‘rinli bo‘lishini ta’minlaydi, ya’ni $\{s_n\}$ ketma-ketlik chekli limitga ega va demak $\sum_{n=1}^{\infty} u_n$ qator yaqinlashuvchi bo‘ladi.◄

11.5. Musbat hadli qatorlar

Barcha hadi musbat sonlardan iborat, ya’ni $u_n > 0, n \in \mathbb{N}$ bo‘lgan $\sum_{n=1}^{\infty} u_n$ qatorlar musbat hadli qatorlar deyiladi.

Musbat hadli $\sum_{n=1}^{\infty} u_n$ qatorning $\{s_n\}$ qismiy yig‘indilari ketma-ketligi o‘suvchi bo‘ladi, chunki $s_{n+1} - s_n = u_{n+1} > 0, n \in \mathbb{N}$. Veyyershtrass alomatiga ko‘ra $\{s_n\}$ ketma-ketlik chekli limitga ega va demak $\sum_{n=1}^{\infty} u_n$ qator qator yaqinlashuvchi bo‘lishligi uchun $\{s_n\}$ ketma-ketlik yuqoridan chegaralangan bo‘lishi zarur va yetarli.

11.6-Teorema (taqqoslash alomati). u_n va v_n hadlari musbat sonlardan iborat

$$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n \tag{11.8}$$

va

$$v_1 + v_2 + \cdots + v_n + \cdots = \sum_{n=1}^{\infty} v_n \tag{11.9}$$

musbat hadli qatorlar berilgan bo‘lsin. Agar barcha n tartib raqamlari uchun $u_n \leq v_n$ tengsizlik o‘rinli bo‘lsa, $\sum_{n=1}^{\infty} v_n$ qatorning yaqinlashishidan $\sum_{n=1}^{\infty} u_n$ qatorning ham yaqinlashishi, $\sum_{n=1}^{\infty} u_n$ qatorning uzoqlashishidan $\sum_{n=1}^{\infty} v_n$ qatorning ham uzoqlashishi kelib chiqadi.

►Berilgan qatorlarning qismiy yig‘indilarini tuzamiz:

$$U_n = u_1 + u_2 + \cdots + u_n, \qquad V_n = v_1 + v_2 + \cdots + v_n.$$

Teoremaning $u_n \leq v_n$ shartidan barcha $n = 1,2, \dots$ uchun $U_n \leq V_n$ ekanligi kelib chiqadi.

1) $\sum_{n=1}^{\infty} v_n$ qator yaqinlashuvchi, ya’ni qismiy yig‘indilar ketma-ketligi chekli $\lim_{n\to\infty} V_n = V$ limitga ega. Berilgan qatorlarning hadlari musbat bo‘lganligi uchum $0 < V_n \leq V$ tengsizlik o‘rinli bo‘ladi, bundan esa $u_n \leq v_n$ tengsizlikga ko‘ra $0 < U_n < V$ tengsizlikga ega bo‘lamiz. Shunday qilib $\sum_{n=1}^{\infty} u_n$ qatorning $\{U_n\}$ qismiy yig‘indilari ketma-ketligi yuqoridan chegaralangan va o‘suvchi. Demak $\{U_n\}$ qismiy yig‘indilari ketma-ketligi va demak $\sum_{n=1}^{\infty} u_n$ qator yaqinlashuvchi.

2) $\sum_{n=1}^{\infty} u_n$ qator uzoqlashuvchi bo‘lsin. Barcha hadlar $u_n > 0$ bo‘lganligi uchun $\{U_n\}$ ketma-ketlik o‘suvchi va demak $\lim_{n\to\infty} U_n = +\infty$ bo‘ladi. $U_n \leq V_n$

tengsizlikdan $\lim\limits_{n\to\infty} V_n = +\infty$ tenglik kelib chiqadi, ya'ni $\sum\limits_{n=1}^{\infty} v_n$ qator uzoqlashuvchi. ◄

11.9-Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 5}$$

qatorning yaqinlashishini tekshiring.

►11.5-Misolda qaralgan $\sum\limits_{n=1}^{\infty} q^{n-1}$ qatorda $q = 1/2$ bo'lsa, yaqinlashuvchi

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

qatorni hosil qilamiz.

Berilgan qatorni bu qator bilan taqqoslaymiz. Ularning hadlari uchun $n = 1,2, \ldots$ bo'lganda

$$\frac{1}{2^n + 5} < \frac{1}{2^n}$$

tengsizliklar o'rinli. Shuning uchun 11.6-Teoremaga ko'ra berilgan qator ham yaqinlashuvchi.◄

11.10-Misol. Garmonik qator deb ataluvchi

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

qator yaqinlashishini teksiring.

►Garmonik qatorning uchinchi va to'rtinchi hadlarini $1/4$ bilan, keyingi to'rtta hadini $1/8$ bilan, keyingi 8 ta hadini $1/16$ bilan va hokazo almashtirishlarni bajaramiz. Natijada

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \underbrace{\frac{1}{16} + \cdots \frac{1}{16}}_{8 \text{ ta had}} + \underbrace{\frac{1}{32} + \cdots + \frac{1}{32}}_{16 \text{ ta had}} + \cdots$$

qatorni hosil qilamiz. Bu qatorning har bir hadi garmonik qatorning mos hadidan katta emas. Shuning uchun bu qatorning uzoqlashuvchi ekanligini isbotlasak 11.6-Teoremaga ko'ra garmonik qator ham uzoqlashuvchi bo'ladi. Haqiqattan ham hosil qilingan qatorning bir xil qiymatli hadlarini guruhlaymiz. Bunda k −guruhda 2^k ta had bo'lib, hadlarning maxraji 2^{k+1} ga teng. Shuning uchun bunday guruhning yig‘indisi $\frac{2^k}{2^{k+1}}=\frac{1}{2}$ bo‘ladi, ya’ni hosil qilingan qator

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}+\cdots$$

ko‘rinishda bo‘ladi. Ravshanki bu so‘ngi qator uzoqlashuvchi va demak garmonik qator ham uzoqlashuvchi bo‘ladi.◄

11.7-Teorema (limitik taqqoslash alomati). Musbat hadli (11.8) va (11.9) qatorlar berilgan bo‘lsin. Agar noldan farqli

$$\lim_{n\to\infty}\frac{u_n}{v_n}=r\ \ (0<r<+\infty)$$

chekli limit mavjud bo‘lsa, (11.8) va (11.9) qatorlar bir vaqtda yaqinlashadi yoki uzoqlashadi.

►Ketma-ketlik limiti ta’rifiga ko‘ra ixtiyoriy $\varepsilon>0$ son uchun shunday N tartib raqami topilib, $n>N$ tengsizlikni qanoatlantiruvchi barcha n larda

$$\left|\frac{u_n}{v_n}-r\right|<\varepsilon$$

yoki

$$(r-\varepsilon)v_n<u_n<(r+\varepsilon)v_n \tag{11.10}$$

tengsizlik o‘rinli bo‘ladi.

Agar (11.8) qator yaqinlashuvchi bo‘lsa, (11.10) chap tengsizligi va 11.6-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty}(r-\varepsilon)v_n$ qator ham yaqinlashuvchi bo‘ladi. Bundan esa o‘z navbatida yaqinlashuvchi qatorlarning xossalariga ko‘ra (11.9) qatorning yaqinlashuvchiligi kelib chiqadi.

Agar (11.8) qator uzoqlashuvchi bo‘lsa, (11.10) o‘ng tengsizligi va 11.6-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty}(r+\varepsilon)v_n$ qator va demak $\sum\limits_{n=1}^{\infty}v_n$ qator ham uzoqlashuvchi bo‘ldi.

Xuddi shu singari (11.9) qatorning yaqinlashuvchi (uzoqlashuvchi) ekanligidan (11.8) qatorning yaqinlashuvchi (uzoqlashuvchi) ekanligi ko‘rsatiladi. ◄

11.11-Misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{2n+3}{5n^2+3n+1}$$

qator yaqinlashishini tekshiring.

►Bu qatorni garmonik qator bilan taqqoslaymiz. Ularning umumiy hadlari

$$u_n = \frac{1}{n}, \qquad v_n = \frac{2n+3}{5n^2+3n+1}$$

bo‘ladi. Ular nisbatining limitini hisoblaymiz:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n+3}{5n^2+3n+1}} = \lim_{n\to\infty} \frac{5n^2+3n+1}{n(n+3)} =$$

$$= \lim_{n\to\infty} \frac{5+\frac{3}{n}+\frac{1}{n^2}}{1+\frac{3}{n^2}} = 5.$$

Garmonik qator uzoqlashuvchi bo‘lganligi uchun 11.7-Teoremaga ko‘ra berilgan qator ham uzoqlashuvchi.◄

11.12-Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{5n-1}{2n^3+5n^2+n-2}$$

qator yaqinlashishini tekshiring.

►Berilgan qatorni 11.6-Misolda qaralgan yaqinlashuvchi va umumiy hadi $u_n = 1/n(n+1)$ bo‘lgan qator bilan taqqoslaymiz. 11.7-Teoremada ko‘rsatilgan limitni hisoblaymiz:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{5n-1}{2n^3+5n^2+n-2}} = \lim_{n\to\infty} \frac{2n^3+5n^2+n-2}{n(n+1)(5n-1)} =$$

$$= \lim_{n\to\infty} \frac{2n^3+5n^2+n-2}{5n^3+4n^2-n} = \lim_{n\to\infty} \frac{2+\frac{5}{n}+\frac{1}{n^2}-\frac{2}{n^3}}{5+\frac{4}{n}-\frac{1}{n^3}} = \frac{2}{5}.$$

11.7-Teoremaga ko‘ra berilgan qator yaqinlashuvchi.◄

11.8-Teorema (qator yaqinlashishining Makloren-Koshi integral alomati). Musbat hadli

$$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n \qquad (11.11)$$

qatorning hadlari o‘smasin:

$$u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \qquad (11.12)$$

$[1;+\infty)$ oraliqda aniqlangan uzluksiz $f(x)$ funksiya o‘smasin va

$$f(1)=u_1,\, f(2)=u_2,\, \dots,\, f(n)=u_n,\, \dots \tag{11.13}$$

tengliklar o‘rinli bo‘lsin. U holda (11.11) qatorning yaqinlashishi uchun

$$\int_1^{\infty} f(x)dx \tag{11.14}$$

xosmas integralning yaqinlashishi zarur va yetarli.

►(11.12) tengsizliklar va (11.13) tengliklarga ko‘ra ixtiyoriy $x \in [n-1;n]$ nuqtalarda

$$u_{n-1} \geq f(x) \geq u_n$$

tengsizliklar o‘rinli bo‘ladi. Bu tengsizliklarni hadma-had $[n-1;n]$ kesmada integrallasak

$$u_{n-1}\int_{n-1}^{n} dx \geq \int_{n-1}^{n} f(x)dx \geq u_n \int_{n-1}^{n} dx$$

yoki

$$u_{n-1} \geq \int_{n-1}^{n} f(x)dx \geq u_n \tag{11.15}$$

tengsizliklarga ega bo‘lamiz.

(11.15) tengsizliklarni har bir n uchun yozamiz:

$$u_1 \geq \int_1^2 f(x)dx \geq u_2,$$

$$u_2 \geq \int_2^3 f(x)dx \geq u_3,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_{n-1} \geq \int_{n-1}^{n} f(x)dx \geq u_n.$$

Bu tengsizliklarni hadma-had qo‘shsak

$$s_n - u_n \geq I_n \geq s_n - u_1 \tag{11.16}$$

tengsizliklar hosil bo‘ladi, bu yerda s_n berilgan (11.11) qatorning qismiy yig‘indisi va

$$I_n = \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_{n-1}^{n} f(x)dx = \int_1^n f(x)dx.$$

(11.16) tengsizliklardan

$$I_n < s_n \leq I_n + u_1 \tag{11.17}$$

tengsizliklarni yozish mumkin.

Zarurligi. (11.11) qator yaqinlashuvchi, ya'ni chekli $\lim_{n\to\infty} s_n = s$ limit mavjud bo'lsin. U holda (11.17) chap tengsizligidan $I_n < s$ ekanligi ravshan. Aniqlanishiga ko'ra $\{I_n\}$ o'suvchi va u yuqoridan chegaralangan. Veyyershtrass teoremasiga ko'ra chekli $\lim_{n\to\infty} I_n = I$ limit mavjud, ya'ni (11.14) xosmas integral yaqinlashuvchi.

Yetarliligi. (11.14) xosmas integral yaqinlashuvchi, ya'ni chekli $\lim_{n\to\infty} I_n = I$ limit mavjud bo'lsin. (11.17) o'ng tengsizlikdan $s_n \leq I + u_1$ ekanligi kelib chiqadi. (11.11) musbat hadli qator bo'lganligi uchun $\{s_n\}$ o'suvchi va yuqoridan chegaralangan. Bu yerda ham Veyyershtrass teoremasiga ko'ra chekli $\lim_{n\to\infty} s_n = s$ limit mavjud, ya'ni (11.11) qator yaqinlashuvchi. ◄

11.13-Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

qator yaqinlashishini tekshiring.

►Qator yaqinlashishining Makloren-Koshi alomatidan foydalanamiz. Buning uchun

$$I = \int_1^{\infty} \frac{1}{x^\alpha} dx$$

xosmas integralni qaraymiz. 6.10-Misolda bu xosmas integralning $\alpha \leq 1$ bo'lganda uzoqlashuvchiligini va $\alpha > 1$ bo'lganda yaqinlashuvchiligini ko'rgan edik. Shuning uchun 11.8-Teoremaga ko'ra berilgan qator $\alpha > 1$ bo'lganda yaqinlashuvchi va $\alpha \leq 1$ bo'lgannda esa uzoqlashuvchi bo'ladi. ◄

11.14-Misol. Ushbu

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

qator yaqinlashishini tekshiring.

►Bu yerda ham qator yaqinlashishining Makloren-Koshi alomatidan foydalanamiz. Buning uchun

$$I = \int_2^{\infty} \frac{1}{x \ln x} dx$$

xosmas integralni qaraymiz.

$$I = \lim_{b\to+\infty} \int_2^{b} \frac{1}{x \ln x} dx = \lim_{b\to+\infty} \ln\ln x|_2^b = \lim_{b\to+\infty} (\ln\ln b - \ln\ln 2) = \infty,$$

ya’ni xosmas integral va demak berilgan qator ham uzoqlashuvchi.◄

11.15-Misol. $\sum_{n=1}^{\infty}\frac{1}{e^n}$ qator yaqinlashishini tekshiring.

►6.7-Misolda $\int_a^{+\infty} e^{-x}dx$ xosmas integral o‘rganilgan edi va uning yaqinlashuvchi ekanligini aniqlagan edik. U holda 11.8-Teoremaga ko‘ra berilgan qator ham yaqinlashuvchi bo‘ladi.◄

11.9-Teorema (qator yaqinlashishining Dalamber alomati). Musbat hadli

$$\sum_{n=1}^{\infty} u_n$$

qator berilgan bo‘lsin. Agar

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n}=r$$

chekli limit mavjud bo‘lsa, u holda $r<1$ bo‘lganda berilgan qator yaqinlashuvchi va $r>1$ bo‘lganda qator uzoqlashuvchi bo‘ladi. $r=1$ bo‘lganda masala ochiq qoladi, ya’ni bu holda qator yaqinlashishi ham uzoqlashishi ham mumkin.

► $\lim_{n\to+\infty}\frac{u_{n+1}}{u_n}=r$ bo‘lsin. U holda limitning ta’rifiga ko‘ra ixtiyoriy $\varepsilon>0$ son uchun shunday N_0 tarib raqami topilib, $n>N_0$ tengsizlikni qanoatlantiruvchi barcha n uchun

$$\left|\frac{u_{n+1}}{u_n}-r\right|<\varepsilon$$

yoki

$$-\varepsilon<\frac{u_{n+1}}{u_n}-r<\varepsilon$$

tengsizligi o‘rinli bo‘ladi. Bundan esa

$$r-\varepsilon<\frac{u_{n+1}}{u_n}<r+\varepsilon$$

tengsizlikga ega bo‘lamiz. So‘ngi tengsizlikdan esa

$$(r-\varepsilon)u_n<u_{n+1}<(r+\varepsilon)u_n \tag{11.18}$$

tengsizlini yozish mumkin.

1) $r<1$ bo‘lsin. $\varepsilon>0$ sonni $r+\varepsilon=q<1$ bo‘ladigan qilib tanlaymiz. U holda ixtiyoriy k natural son uchun (11.18) ning o‘ng tomondagi tengsizliklarga ko‘ra

$$u_{N_0+1} < qu_{N_0},\, u_{N_0+2} < qu_{N_0+1} < q^2 u_{N_0},\, \dots ,\, u_{N_0+k} < q^k u_{N_0} \quad (11.19)$$

tengsizlilkar o‘rinli bo‘ladi. Yangi $\sum\limits_{n=1}^{\infty} v_n$ qator tuzamiz, bunda

$$v_1 = u_1, v_2 = u_2, \dots, v_{N_0} = u_{N_0}, v_{N_0+1} = qu_{N_0}, \dots, v_{N_0+k} = q^k u_{N_0}, \dots$$

deb olamiz.

$q < 1$ ekanligidan $\sum\limits_{k=1}^{\infty} q^k u_{N_0} = u_{N_0} \sum\limits_{k=1}^{\infty} q^k$ qator yaqinlashuvchi bo‘ladi, shuning uchun

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{N_0} u_n + \sum_{k=1}^{\infty} q^k u_{N_0}$$

qator ham yaqinlashuvchi bo‘ladi. Bundan tashqari (11.19) tengsizliklardan

$$u_1 \le v_1,\, u_2 \le v_2,\, \dots,\, u_{N_0} \le v_{N_0},\, u_{N_0+1} < v_{N_0+1}, \dots$$

tengsizliklarni yozish mumkin. U holda 11.6-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty} u_n$ qator ham yaqinlashuvchi bo‘ladi.

1) $r > 1$ bo‘lsin. $\varepsilon > 0$ sonni $r - \varepsilon = q > 1$ bo‘ladigan qilib tanlaymiz. U holda ixtiyoriy k natural son uchun (11.18) ning chap tomondagi tengsizliklarga ko‘ra

$$qu_{N_0} < u_{N_0+1},\, q^2 u_{N_0} < u_{N_0+2},\, \dots ,\, q^k u_{N_0} < u_{N_0+k} \quad (11.20)$$

tengsizlilkar o‘rinli bo‘ladi. Yangi $\sum\limits_{n=1}^{\infty} v_n$ qator tuzamiz, bunda

$$v_1 = u_1, v_2 = u_2, \dots, v_{N_0} = u_{N_0}, v_{N_0+1} = qu_{N_0}, \dots, v_{N_0+k} = q^k u_{N_0}, \dots$$

deb olamiz.

$q > 1$ ekanligidan $\sum\limits_{k=1}^{\infty} q^k u_{N_0} = u_{N_0} \sum\limits_{k=1}^{\infty} q^k$ qator uzoqlashuvchi bo‘ladi, shuning uchun

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{N_0} u_n + \sum_{k=1}^{\infty} q^k u_{N_0}$$

qator ham uzoqlashuvchi bo‘ladi. Bundan tashqari (11.20) tengsizliklardan

$$u_1 \ge v_1,\, u_2 \ge v_2,\, \dots,\, u_{N_0} \ge v_{N_0},\, u_{N_0+1} > v_{N_0+1}, \dots$$

tengsizliklarni yozish mumkin. U holda 11.6-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty} u_n$ qator ham uzoqlashuvchi bo‘ladi.◄

11.16-Misol. $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{2^n (n+1)!}$ qator yaqinlashishini tekshiring.

►Dalamber alomatiga ko‘ra tekshiramiz. Bu yerda

$$u_n = \frac{3^n}{2^n(n+1)!}, u_{n+1} = \frac{3^{n+1}}{2^{n+1}(n+2)!}.$$

$\frac{u_{n+1}}{u_n}$ nisbatning limitini hisoblaymiz:

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{\frac{3^{n+1}}{2^{n+1}(n+2)!}}{\frac{3^n}{2^n(n+1)!}} = \lim_{n\to+\infty}\frac{3^{n+1}2^n(n+1)!}{3^n2^{n+1}(n+2)!} =$$

$$= \lim_{n\to+\infty}\frac{3}{2(n+2)} = 0 < 1,$$

demak 11.9-Teoremaga ko‘ra berilgan qator yaqinlashuvchi.◄

11.17-Misol. $\sum\limits_{n=1}^{\infty}\frac{n^n}{2^n n!}$ qator yaqinlashishini tekshiring.

► Dalamber alomatiga ko‘ra tekshiramiz. Bu yerda

$$u_n = \frac{n^n}{2^n n!}, u_{n+1} = \frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}.$$

$\frac{u_{n+1}}{u_n}$ nisbatning limitini hisoblaymiz:

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{\frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}}{\frac{n^n}{2^n n!}} = \lim_{n\to+\infty}\frac{(n+1)^{n+1}2^n n!}{n^n2^{n+1}(n+1)!} =$$

$$= \lim_{n\to+\infty}\frac{(n+1)^{n+1}}{n^n 2(n+1)} = \frac{1}{2}\lim_{n\to+\infty}\frac{(n+1)^n}{n^n} = \frac{1}{2}\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{2} > 1,$$

demak 11.9-Teoremaga ko‘ra berilgan qator uzoqlashuvchi.◄

11.10-Teorema (qator yaqinlashishining Koshi alomati). Musbat hadli

$$\sum_{n=1}^{\infty} u_n$$

qator berilgan bo‘lsin. Agar

$$\lim_{n\to+\infty}\sqrt[n]{u_n} = r$$

chekli limit mavjud bo‘lsa, u holda $r < 1$ bo‘lganda berilgan qator yaqinlashuvchi va $r > 1$ bo‘lganda qator uzoqlashuvchi bo‘ladi. $r = 1$ bo‘lganda masala ochiq qoladi, ya’ni bu holda qator yaqinlashishi ham uzoqlashishi ham mumkin.

► $\lim\limits_{n\to+\infty}\sqrt[n]{u_n}=r$ bo‘lsin. U holda limitning ta’rifiga ko‘ra ixtiyoriy $\varepsilon>0$ son uchun shunday N_0 tarib raqami topilib, $n>N_0$ tengsizlikni qanoatlantiruvchi barcha n uchun

$$\left|\sqrt[n]{u_n}-r\right|<\varepsilon$$

yoki

$$-\varepsilon<\sqrt[n]{u_n}-r<\varepsilon$$

tengsizligi o‘rinli bo‘ladi. Bundan esa

$$r-\varepsilon<\sqrt[n]{u_n}<r+\varepsilon \qquad (11.21)$$

tengsizlikga ega bo‘lamiz.

1) $r<1$ bo‘lsin. $\varepsilon>0$ sonni $r+\varepsilon=q<1$ bo‘ladigan qilib tanlaymiz. U holda ixtiyoriy k natural son uchun (11.21) ning o‘ng tomondagi tengsizliklarga ko‘ra

$$\sqrt[N_0]{u_{N_0}}<q,\ \sqrt[N_0+1]{u_{N_0+1}}<q,\ldots,\ \sqrt[N_0+k]{u_{N_0+k}}<q,\ldots$$

yoki

$$u_{N_0+1}<q^{N_0+1}, u_{N_0+2}<q^{N_0+2},\ldots, u_{N_0+k}<q^{N_0+k},\ldots \qquad (11.22)$$

tengsizliklar o‘rinli bo‘ladi.

Yangi $\sum\limits_{n=1}^{\infty}v_n$ qator tuzamiz, bunda

$$v_1=u_1, v_2=u_2,\ldots,v_{N_0}=u_{N_0}, v_{N_0+1}=q^{N_0+1},\ldots,v_{N_0+k}=q^{N_0+k},\ldots$$

deb olamiz.

$q<1$ ekanligidan $\sum\limits_{k=1}^{\infty}q^{N_0+k}$ qator yaqinlashuvchi bo‘ladi, shuning uchun

$$\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{N_0}u_n+\sum_{k=1}^{\infty}q^{N_0+k}$$

qator ham yaqinlashuvchi bo‘ladi. Bundan tashqari (11.22) tengsizliklardan

$$u_1\le v_1,\ u_2\le v_2,\ \ldots,\ u_{N_0}\le v_{N_0},\ u_{N_0+1}<v_{N_0+1},\ldots$$

tengsizliklarni yozish mumkin. U holda 11.6-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty}u_n$ qator ham yaqinlashuvchi bo‘ladi.

1) $r>1$ bo‘lsin. $\varepsilon>0$ sonni $r-\varepsilon=q>1$ bo‘ladigan qilib tanlaymiz. U holda ixtiyoriy k natural son uchun (11.21) ning chap tomondagi tengsizliklarga ko‘ra

$$q < \sqrt[N_0+1]{u_{N_0+1}},\, q < \sqrt[N_0+2]{u_{N_0+2}},\ldots,\, q < \sqrt[N_0+k]{u_{N_0+k}},\ldots$$

yoki

$$q^{N_0+1} < u_{N_0+1},\, q^{N_0+2} < u_{N_0+2},\ldots,\, q^{N_0+k} < u_{N_0+k},\ldots \qquad (11.23)$$

tengsizliklar o‘rinli bo‘ladi.

Bu yerda ham hadlari

$$v_1 = u_1, v_2 = u_2, \ldots, v_{N_0} = u_{N_0}, v_{N_0+1} = q^{N_0+1}, \ldots, v_{N_0+k} = q^{N_0+k}, \ldots$$

tengliklar bilan aniqlanadigan yangi $\sum_{n=1}^{\infty} v_n$ qator tuzamiz.

$q > 1$ ekanligidan $\sum_{k=1}^{\infty} q^{N_0+k}$ qator uzoqlashuvchi bo‘ladi, shuning uchun

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{N_0} u_n + \sum_{k=1}^{\infty} q^{N_0+k}$$

qator ham uzoqlashuvchi bo‘ladi. Bundan tashqari (11.23) tengsizliklardan

$$v_1 \le u_1,\, v_2 \le u_2,\, \ldots,\, v_{N_0} \le u_{N_0},\, v_{N_0+1} < u_{N_0+1},\ldots$$

tengsizliklarni yozish mumkin. U holda 11.6-Teoremaga ko‘ra $\sum_{n=1}^{\infty} u_n$ qator ham uzoqlashuvchi bo‘ladi.◄

11.18-Misol. $\sum_{n=1}^{\infty} \left(\frac{4n-1}{9n+1}\right)^n$ qator yaqinlashishini tekshiring.

►Qator yaqinlashishini Koshi alomatiga ko‘ra tekshiramiz. Bu yerda $u_n = \left(\frac{4n-1}{9n+1}\right)^n$ va $\lim\limits_{n\to+\infty} \sqrt[n]{u_n}$ limitni hisoblaymiz:

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{4n-1}{9n+1}\right)^n} = \lim_{n\to+\infty} \frac{4n-1}{9n+1} = \frac{4}{9} < 1$$

11.10-Teoremaga ko‘ra berilgan qator yaqinlashuvchi.◄

11.19-Misol. $\sum_{n=1}^{\infty} 3^n e^{-n}$ qator yaqinlashishini tekshiring.

►Qator yaqinlashishini Koshi alomatiga ko‘ra tekshiramiz. Bu yerda $u_n = 3^n e^{-n}$ va $\lim\limits_{n\to+\infty} \sqrt[n]{u_n}$ limitni hisoblaymiz:

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{3^n e^{-n}} = \frac{3}{e} > 1$$

11.10-Teoremaga ko‘ra berilgan qator uzoqlashuvchi.◄

Yaqinlashuvchi musbat hadli qatorlarning muhim bir xossasi ushbu teoremada keltirilgan.

11.11-Teorema (Dirixle). $\sum_{n=1}^{\infty} u_n$ musbat hadli (U) qator hadlarining o‘rinlarini ixtiyoriy ravishda almashtirishdan yangi $\sum_{n=1}^{\infty} v_n$ (V) qator hosil qilingan bo‘lsin. Agar berilgan (U) qator yaqinlashuvchi va uning yig‘indisi s bo‘lsa, yangi hosil qilingan (V) qator ham yaqinlashadi va uning ham yig‘indisi s bo‘ladi.

► (V) qatorning

$$V_n = v_1 + v_2 + \cdots + v_n$$

qismiy yig‘indisini qaraymiz. Bu yig‘indining har bir hadi (U) qatorga ham kiradi. m sonini shunday katta qilib tanlaymizki, natijada V_n qismiy yigindining barcha hadlari (U) qatorning

$$U_m = u_1 + u_2 + \cdots + u_m$$

qismiy yig‘indisiga kirsin. U holda qatorlar musbat hadli bo‘lganligi uchun

$$V_n \leq U_m \leq s$$

tengsizliklarning o‘rinli bo‘lishi va $\{V_n\}$ ketma-ketlik monoton o‘sishi ravshan. Shuning uchun 1.12-Teoremaga ko‘ra $\{V_n\}$ ketma-ketlik va demak (V) qator yaqinlashuvchi bo‘ldi. Bundan tashqari

$$s' = \lim_{n\to+\infty} V_n \leq s$$

tengsizlik o‘rinli. O‘z navbatida (U) qatorni (V) qator hadlarining o‘rnini almashtirishdan hosil qilingan qator deb qarash mumkin. Yuqoridagi singari mulohazalar yuritib

$$s \leq s'$$

tengsizlikni olish mumkin, bundan esa $s = s'$ tenglikga kelamiz.◄

11.6. O‘zgaruvchan ishorali qatorlar

Agar qator hadlarining orasida turli ishorali sonlar mavjud bo‘lsa, bunday qatorlarni *o‘zgaruvchan ishorali qatorlar* deb ataymiz. Mazkur bo‘limda hadlari haqiqiy sonlardan iborat o‘zgaruvchan ishorali qatorlar bilan bir qatorda kompleks hadli qatorlarni ham qaraymiz. O‘zgaruvchan ishorali

$$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n \qquad (11.24)$$

qator va uning hadlarining absolyut qiymatlaridan tuzilgan

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots = \sum_{n=1}^{\infty} |u_n| \qquad (11.25)$$

qatorlar berilgan bo‘lsin. (11.25) qator musbat hadli qator ekanligi ravshan. Biz bunday qatorlarni yuqorida ko‘rdik.

11.6-Ta’rif. Agar qator hadlarining absolyut qiymatlaridan tuzilgan (11.25) qator yaqinlashuvchi bo‘lsa, berilgan (11.24) qator absolyut yaqinlashuvchi deyiladi.

Agar $\sum_{n=1}^{\infty} u_n$ qator kompleks qator, ya’ni $u_n = a_n + ib_n$, bu yerda $a_n, b_n \in \mathbb{R}, n \in \mathbb{N}$ bo‘lsa, $|u_n| = \sqrt{a_n^2 + b_n^2}$ va

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$$

ekanligini eslatib o‘tamiz.

11.12-Teorema. Kompleks

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (a_n + ib_n)$$

qator absolyut yaqinlashuvchi bo‘lishligi uchun $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ haqiqiy hadli qatorlarning absolyut yaqinlashishi zarur va yetarli.

►**Zarurligi.** $\sum_{n=1}^{\infty} u_n$ kompleks qator yaqinlashuvchi bo‘lsin. U holda ixtiyoriy $n \in \mathbb{N}$ uchun

$$|a_n| \le \sqrt{a_n^2 + b_n^2} = |u_n|, |b_n| \le \sqrt{a_n^2 + b_n^2} = |u_n|$$

tengsizliklar o‘rinli bo‘ladi. U holda 11.6-Teoremaga ko‘ra $\sum_{n=1}^{\infty} |u_n|$ qatorning yaqinlashishidan $\sum_{n=1}^{\infty} |a_n|$ va $\sum_{n=1}^{\infty} |b_n|$ qatorlarning ham yaqinlashishi kelib chiqadi.

Yetarliligi. $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ haqiqiy hadli qatorlar absolyut yaqinlashuvchi bo‘lsin. Ushbu

$$a_n^2 + b_n^2 = |a_n|^2 + |b_n|^2 \le |a_n|^2 + 2|a_n||b_n| + |b_n|^2 = (|a_n| + |b_n|)^2$$

tengsizlikdan

$$|a_n + ib_n| = \sqrt{a_n^2 + b_n^2} \le |a_n| + |b_n|$$

tengsizlikni yozish mumkin. Yana 11.6-Teoremaga ko‘ra $\sum_{n=1}^{\infty}|a_n|$ va $\sum_{n=1}^{\infty}|b_n|$ qatorlarning yaqinlashishidan $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left||a_n + ib_n|\right|$ qatorning ham yaqinlashishi kelib chiqadi.◄

11.13-Teorema. Har qanday absolyut yaqinlashuvchi qator yaqinlashuvchi hamdir.

► $\sum_{n=1}^{\infty}u_n$ qator absolyut yaqinlashuvchi va demak $\sum_{n=1}^{\infty}|u_n|$ qator yaqinlashuvchi bo‘lsin. U holda 11.5-Teoremada keltirilgan Koshi kriteriyasiga ko‘ra ixtiyoriy $\varepsilon > 0$ soni uchun shunday $N = N(\varepsilon)$ tartib raqami topilib, ixtiyoriy $n > N$ va $p = 0, 1, 2, \dots$ qiymatlarda

$$|u_n| + |u_{n+1}| + \cdots + |u_{n+p}| < \varepsilon \qquad (11.26)$$

tengsizlik o‘rinli bo‘ladi. Bundan tashqari haqiqiy sonlar ustida arifmetik amallarning ma’lum xossalariga va kompleks sonlar uchun (1.21) tengsizliklarga ko‘ra

$$|u_n + u_{n+1} + \cdots + u_{n+p}| \le |u_n| + |u_{n+1}| + \cdots + |u_{n+p}|$$

tengsizlikni yozish mumkin. Bundan esa (11.26) ga ko‘ra $u_n + u_{n+1} + \cdots + u_{n+p}$ ifoda uchun Koshi kriteriyasining o‘rinli ekanligiga kelamiz, ya’ni berilgan $\sum_{n=1}^{\infty}u_n$ qator yaqinlashuvchi.◄

11.20-Misol. $\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n^3}$ qator yaqinlashishini tekshiring.

► Bu qator o‘zgaruvchan ishorali qator. Uning hadlari uchun

$$\left|\frac{\sin n\alpha}{n^3}\right| \le \frac{1}{n^3}$$

tengsizliklar o‘rinli. Bundan tashqari 11.13-Misolda $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator $\alpha > 1$ bo‘lganda yaqinlashuvchi ekanligini ko‘rgan edik. Shuning uchun $\sum_{n=1}^{\infty}\frac{1}{n^3}$ qator yaqinlashuvchi bo‘ladi. U holda 11.6-Teoremaga ko‘ra $\sum_{n=1}^{\infty}\left|\frac{\sin n\alpha}{n^3}\right|$ qator ham va demak 11.13-Teoremaga ko‘ra berilgan qator ham yaqinlashuvchi bo‘ladi.◄

11.14-Teorema. Agar $\sum_{n=1}^{\infty} u_n$ qator absolyut yaqinlashuvchi va uning yig'indisi s hamda, $\sum_{n=1}^{\infty} |u_n|$ qatorning yig'indisi S bo'lsa,

$$|s| \leq S$$

tengsizlik o'rinli bo'ladi.

► $\sum_{n=1}^{\infty} u_n$ qatorning qismiy yig'indilari uchun

$$|s_n| = |u_1 + u_2 + \cdots + u_n| \leq |u_1| + |u_2| + \cdots + |u_n|$$

tengsizlik o'rinli bo'ladi va bu tengsizlikda limitga o'tsak $|s| \leq S$ tengsizlikga ega bo'lamiz.◄

11.7-Ta'rif. Agar berilgan $\sum_{n=1}^{\infty} u_n$ qator yaqinlashuvchi va bu qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{n=1}^{\infty} |u_n|$ qator uzoqlashuvchi bo'lsa, berilgan qator shartli yaqinlashuvchi deyiladi.

11.21-Misol. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ qator yaqinlashishini tekshiring.

► Bu qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{n=1}^{\infty} \frac{1}{n}$ qator (garmonik qator) uzoqlashuvchi ekanligini 11.10-Misolda ko'rgan edik. Bu qatorning o'zi yaqinlashuvchi ekanligini ko'rsatamiz. Qator ko'rinishini o'zgartiramiz:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots =$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots =$$

$$= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1) \cdot 2n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1) \cdot 2n}.$$

Demak hosil qilingan qator umumiy hadi $u_n = \frac{1}{(2n-1)\cdot 2n}$ bo'ladi. Uni biz umumiy hadi $v_n = \frac{1}{n^2}$ bo'lgan $\sum_{n=1}^{\infty} \frac{1}{n^2}$ qator bilan taqqoslaymiz:

$$\lim_{n\to+\infty}\frac{v_n}{u_n}=\lim_{n\to+\infty}\frac{\dfrac{1}{n^2}}{\dfrac{1}{(2n-1)\cdot 2n}}=\lim_{n\to+\infty}\frac{(2n-1)\cdot 2n}{n^2}=$$
$$=\lim_{n\to+\infty}2\left(2-\frac{1}{n}\right)=4.$$
$\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ qator 11.13-Misolga ko‘ra yaqinlashuvchi va demak 11.7-Teoremaga ko‘ra $\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)\cdot 2n}$, ya’ni berilgan $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ qator yaqinlashuvchi bo‘ladi. Shunday qilib, berilgan qator shartli yaqinlashuvchi ekan.◄

11.15-Teorema. Agar ozgaruvchan ishorali $\sum\limits_{n=1}^{\infty}u_n$ (U) qator shartli yaqinlashsa, bu qatorning musbat hadlaridan tuzilgan $\sum\limits_{n=1}^{\infty}v_n$ (V) va manfiy hadlarining absolyut qiymatlaridan tuzilgan $\sum\limits_{n=1}^{\infty}w_n$ (W) qatorlar uzoqlashuvchi bo‘ladi.

►Teskarisidan faraz qilaylik, ya’ni (V) va (W) qatorlar yaqinlashuvchi bo‘lsin. (U) qatorda manfiy hadlarni nollar bilan almashtiramiz. Natijada (V) qatorga nollar qo‘shilgan ($V^{(0)}$) qator hosil bo‘ladi va 11.1-Natijaga ko‘ra u yaqinlashadi. Xuddi shu singari (U) qatorda musbat hadlarni nollar bilan almashtirsak (W) qatorga nollar qo‘shilgan yaqinlashuvchi ($W^{(0)}$) qatorni hosil qilamiz. ($V^{(0)}$) va ($W^{(0)}$) qatorlarning yig‘indisi (U) qator hadlarining absolyut qiymatlaridan tuzilgan qatorga teng. Biroq bu qator 11.4-Teoremaga ko‘ra yaqinlashishi kerak. Bu esa teoremaning berilgan qatorning shartli yaqinlashuvchiligi haqidagi shartga ziddir. Demak (V) va (W) qatorlar bir vaqtda yaqinlashuvchi bo‘lmaydi.

Endi (V) yoki (W) qatorlardan biri yaqinlashsin, masalan (V) qator yaqinlashib, (W) qator uzoqlashsin. U holda yuqorida tuzilgan ($V^{(0)}$) qator ham yaqinlashadi. (U) qatordan ($V^{(0)}$) qatorni ayirsak ($W^{(0)}$) qator hosil bo‘lishi ravshan: (U) – ($V^{(0)}$) = ($W^{(0)}$). Biroq tenglikning chap tomonidagi qator 11.1-Natijaga ko‘ra yaqinlashuvchi. Bu esa ($W^{(0)}$) qatorning uzoqlashuvchiligi haqidagi farazimizga ziddir. Demak, (V) va (W) qatorlarning hech biri yaqinlashuvchi bo‘la olmaydi.◄

11.22-Misol. 11.21-Misolda $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ qatorning shartli yaqinlashishini ko‘rgan edik. Bu qator hadlarining o‘rni almashtirilganda yig‘indining qiymati o‘zgarishini ko‘rsatamiz.

►Qatorning yig‘indisi s bo‘lsin. Qator hadlarining o‘rnini bitta musbat haddan keyin ikkita manfiy had keladigan qilib almashtiramiz:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=$$
$$=1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots=$$
$$=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots=$$
$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}s.$$

Shunday qilib, berilgan qator hadlarining o‘rnini yuqorida keltirilgan usul bilan almashtirilganda qator yig‘indisi ikki barobarga kamayganligini ko‘rdik.◄

Shartli yaqinlashuvchi qatorlarning bunday xossasi quyidagi teoremada keltirilgan.

11.16-Teorema (Riman). Agar qator shartli yaqinlashsa, oldindan tanlanadigan S soni qanday (chekli yoki cheksiz) bo‘lishidan qat’iy nazar qator hadlarining o‘rnini shunday almashtirish mumkinki, natijada hosil bo‘ladigan qatorning yig‘indisi ana shu S soniga teng bo‘ladi.

► Ozgaruvchan ishorali $\sum_{n=1}^{\infty} u_n$ (U) qator shartli yaqinlashsin va S ixtiyoriy chekli son bo‘lsin. 11.15-Teoremaga ko‘ra (V) va (W) qatorlar uzoqlashuvchi bo‘ladi, shuning uchun ulardan yig‘indisi ixtiyoriy sondan oshadigan qilib kerakli sondagi hadlarni olishimiz mumkin.

(U) qator hadlarining o‘rnini quyidagicha almashtiramiz.

Dastlab k_1 sonini shunday tanlaymizki, bunda (V) qator dastlabki k_1 ta hadlari yig‘indisi uchun

$$v_1+v_2+\cdots+v_{k_1-1}<S \text{ va } v_1+v_2+\cdots+v_{k_1-1}+v_{k_1}>S$$

tengsizliklar o‘rinli bo‘lsin. Endi bu yig‘inidan (W) qatorning dastlabki l_1 hadini

$$I_1=v_1+v_2+\cdots+v_{k_1-1}+v_{k_1}-w_1-w_2-\cdots-w_{l_1-1}-w_{l_1}<S,$$
$$v_1+v_2+\cdots+v_{k_1-1}+v_{k_1}-w_1-w_2-\cdots-w_{l_1-1}>S$$

bo‘ladigan qilib ayiramiz. U holda bu ayirmaning S sonidan farqi

$$|I_1 - S| \leq w_{l_1}$$

tengsizlikni qanoatlantiradi. Keyingi bosqichda hosil qilingan I_1 ayirmaga (V) qatorning k_1 hadidan keying hadlarini yig‘indi S sonidan oshadigan qilib qo‘shamiz:

$$I_1 + v_{k_1+1} + \cdots + v_{k_2-1} + v_{k_2} > S,$$

$$I_1 + v_{k_1+1} + \cdots + v_{k_2-1} < S.$$

$I_1 + v_{k_1+1} + \cdots + v_{k_2-1} + v_{k_2}$ ifodadan (W) qatorning navbatdagi hadlarini ayirib I_2 ifodani hosil qilamiz:

$$I_2 = I_1 + v_{k_1+1} + \cdots + v_{k_2-1} + v_{k_2} - w_{l_1+1} - \cdots - w_{l_2-1} - w_{l_2}.$$

Bunda I_2 uchun $I_2 < S$ va $I_2 + w_{l_2} > S$ tengsizliklar o‘rinli va uning S sonidan qarqi

$$|I_2 - S| \leq w_{l_2}$$

tengsizlikni qanoatlantiradi. Bu jarayonni cheksiz davom ettirib

$$I_n = I_{n-1} + v_{k_{n-1}+1} + \cdots + v_{k_n} - w_{l_{n-1}+1} - \cdots - w_{l_n}$$

sonlar ketma-ketligini hosil qilamiz. Bu ketma-ketlik hadlari uchun

$$|I_n - S| \leq w_{l_n} \tag{11.27}$$

tengsizliklar o‘rinli. I_n ketma-ketlik (U) qator hadlarining o‘rinlarini almashtirishdan hosil bo‘lgan biror (I) qatorning qismiy yig‘indilari ketma-ketligi hamdir. Bundan tashqari (U) qator yaqinlashuvchi bo‘lganligi uchun $\lim\limits_{n\to+\infty} u_n = 0$ bo‘ladi. w_{l_n} ketma-ketlik u_n ketma-ketlikning qismiy ketma-ketligi ekankigidan $\lim\limits_{n\to+\infty} w_{l_n} = 0$ kelib chiqadi. Bundan esa o‘z navbatida (11.27) tengsizlikga ko‘ra

$$\lim_{n\to+\infty} I_n = S$$

tenglikni olamiz. Bu esa qator yig‘indisining ta’rifiga ko‘ra (I) qatorning yig‘indisi S soniga tengligini anglatadi.

Agar $S = +\infty$ bo‘lsa, (V) qatordan

$$I_1 = v_1 + v_2 + \cdots + v_{k_1} - w_1 > 1$$

bo‘ladigan qilib k_1 ta hadni olamiz. I_2 ni esa

$$I_2 = I_1 + v_{k_1+1} + \cdots + v_{k_2} - w_2 > 2$$

ko‘rinishda tuzamiz. Bu jarayonni ham cheksiz davom ettirib

$$I_n = I_{n-1} + v_{k_{n-1}+1} + \cdots + v_{k_n} - w_n > n$$

sonlar ketma-ketligini hosil qilamiz. Bu yerda ham I_n ketma-ketlik (U) qator hadlarining o‘rinlarini almashtirishdan hosil bo‘lgan biror (I) qator qismiy yig‘indilari ketma-ketligidan iborat va ular uchun $I_n > n$ ekanligidan

$$\lim_{n\to+\infty} I_n = +\infty$$

tenglik kelib chiqadi, bu esa (I) qator yig‘indisi $+\infty$ ga tengligini anglatadi.

$S = -\infty$ bo‘lgan holda ham xuddi shu singari yig‘indisi $-\infty$ bo‘lgan qator tuziladi. ◄

11.8-Ta’rif. Ushbu

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1}u_n + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}u_n \quad (11.28)$$

qator ishoralari navbatlashuvchi qator deb ataladi, bu yerda $u_n > 0$.

11.17-Teorema (Leybnis). Agar ishoralari navbatlashuvchi (11.27) qator hadlari absolyut qiymati bo‘yicha monoton kamaysa:

$$u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots$$

va n cheksiz o‘sganda u_n umumiy had nolga intilsa: $\lim\limits_{n\to+\infty} u_n = 0$, u holda bu qator yaqinlashadi va qator yig‘indisi s uchun

$$0 < s < u_1 \quad (11.29)$$

tengsizlik o‘rinli.

►$2n$ ta had qatnashgan qismiy yig‘indini tuzamiz:

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}).$$

Teorema shartiga ko‘ra qator hadlari absolyut qiymati bo‘yicha monoton kamaydi, shuning uchun s_{2n} qismiy yig‘indida barcha qavslardagi ayirmalar nomanfiy. Bu esa n o‘sganda s_{2n} ketma-ketlik ham o‘sishini anglatadi. Endi s_{2n} qismiy yig‘indining ko‘rinishini o‘zgartiramiz:

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n}.$$

Bu yerda ham barcha qavslardagi ayirmalar nomanfiy, bundan esa $s_{2n} \leq u_1$ ekanligi kelib chiqadi. Demak $\{s_{2n}\}$ ketma-ketlik monoton o‘sadi va u yuqoridan chegaralangan. Shuning uchun 1.12-Teoremaga ko‘ra bu ketma-ketlik chekli limitga ega:

$$\lim_{n\to+\infty} s_{2n} = s \quad (11.30)$$

va bu limit uchun (11.29) tengsizliklarning o‘rinli bo‘lishi ravshan. Qismiy yig‘indilarning aniqlanishiga ko‘ra $s_{2n+1} = s_{2n} + u_{2n+1}$, shuning uchun $\lim\limits_{n\to+\infty} u_n = 0$ tenglikni inobatga olsak

$$\lim_{n\to+\infty} s_{2n+1} = \lim_{n\to+\infty}(s_{2n} + u_{2n+1}) = \lim_{n\to+\infty} s_{2n} + \lim_{n\to+\infty} u_{2n+1} = s$$

tenglikga ega bo‘lamiz. Bu esa (11.30) tenglik bilan birgalikda

$$\lim_{n\to+\infty} s_n = s$$

ekanligini anglatadi.◄

11.23-Misol. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\cdots$ qator yaqinlashishini tekshiring.

►Qatorning umumiy hadi

$$(-1)^{n+1}\frac{2n-1}{2^{n-1}}$$

bo‘ladi, ya’ni $u_n = \frac{2n-1}{2^{n-1}}$ deb olamiz. $\{u_n\}$ ketma-ketlik nolga monoton kamayib yaqinlashishini ko‘rsatamiz:

$$u_n-u_{n+1}=\frac{2n-1}{2^{n-1}}-\frac{2n+1}{2^n}=\frac{4n-2-2n-1}{2^n}=\frac{2n-3}{2^n},$$

bu yerda ixtiyoriy $n\geq 2$ uchun

$$u_n-u_{n+1}=\frac{2n-3}{2^n}>0,$$

ya’ni u_n monoton kamayadi. Endi $\lim\limits_{n\to+\infty} u_n=\lim\limits_{n\to+\infty}\frac{2n-1}{2^{n-1}}=0$ ekanligini ko‘rsatamiz. Buning uchun $f(x)=2x-1$ va $g(x)=2^{x-1}$ funksiyalar $\frac{f(x)}{g(x)}=\frac{2x-1}{2^{x-1}}$ nisbatining $x=+\infty$ dagi limitini hisoblaymiz. Bu limit uchun Lopital qoidasini qo‘llash mumkin:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{2x-1}{2^{x-1}}=\lim_{x\to+\infty}\frac{(2x-1)'}{(2^{x-1})'}=\lim_{x\to+\infty}\frac{2}{2^{x-1}\ln 2}=0.$$

Funksiya limitining Geyne ta’rifiga ko‘ra $\lim\limits_{x\to+\infty}\frac{2x-1}{2^{x-1}}=0$ tenglik $\lim\limits_{n\to+\infty}x_n=+\infty$ munosabatni qanoatlantiruvchi ixtiyoriy $\{x_n\}$, shu lumladan $x_n=n$ ketma-ketlik uchun ham o‘rinli bo‘lishi kerak. Bundan esa $\lim\limits_{n\to+\infty}\frac{2n-1}{2^{n-1}}=0$ tenglikga ega bo‘lamiz.

Shunday qilib, $\{u_n\}$ ketma-ketlik nolga monoton kamayib yaqinlashishar ekan. U holda Leybnis teoremasiga ko‘ra berilgan $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2n-1}{2^{n-1}}$ qator yaqinlashuvchi bo‘ladi.

Biz berilgan qatorning yaqinlashishini Leybnis teoremasini qo‘llab isbotladik. Umuman olganda bu qator absolyut yaqinlashuvchi. Qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{2n-1}{2^{n-1}}$ musbat hadli qator yaqinlashuvchi, chunki

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{\frac{2n+1}{2^n}}{\frac{2n-1}{2^{n-1}}} = \lim_{n\to+\infty} \frac{2^{n-1}(2n+1)}{2^n(2n-1)} = \frac{1}{2} \lim_{n\to+\infty} \frac{2n+1}{2n-1} =$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2} < 1,$$

ya’ni, Dalamber alomatiga ko‘ra $\sum_{n=1}^{\infty} \frac{2n-1}{2^{n-1}}$ qator yaqinlashuvchi. Shuning uchun berilgan qator absolyut yaqinlashuvchi bo‘ladi. ◄

11.24-Misol. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ qator yaqinlashishini tekshiring.

► Biz 11.14-Misolda $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ qator uzoqlashishini ko‘rgan edik. Shuning uchun berilgan qator absolyut yaqinlashuvchi bo‘lmaydi. Biroq u shartli yaqinlashuvchi, chunki berilgan qator uchun Leybnis teoremasining ikkala

$$\frac{1}{2\ln 2} > \frac{1}{3\ln 3} > \cdots > \frac{1}{n \ln n} > \cdots$$

va $\lim_{n\to+\infty} \frac{1}{n \ln n} = 0$ shartlari o‘rinli.◄

11.2-Natija. Leybnis teoremasining shartlarini qanoatlantiruvchi ishoralari navbatlashuvchi $\sum_{n=1}^{\infty} (-1)^{n+1} u_n, (u_n > 0)$ qatorning

$$r_n = (-1)^{n+2} u_{n+1} + (-1)^{n+3} u_{n+2} + \cdots$$

qoldiq qatori uchun

$$|r_n| < |u_{n+1}|$$

tengsizlik o‘rinli bo‘ladi.

► r_n qoldiq qator Leybnis teoramasining shartlarini qanoatlantiradi. Shuning uchun qator yig‘indisining absolyut qiymati uning birinchi hadi absolyut qiymatidan oshmaydi.◄

Mulohaza. Agar qatorning yig‘indisini $\varepsilon > 0$ aniqlikda hisoblash talab qilingan bo‘lsa, $u_{n+1} < \varepsilon$ bo‘ladigan eng kichik n uchun s_n qismiy yig‘indini hisoblash yetarli bo‘ladi.

11.25-Misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ qator yigindisini 0,01 aniqlikda hisoblang.

►Yuqorida keltirilgan mulohazaga ko‘ra $u_{n+1} < 0{,}01$ tengsizlikni qanoatlantiradigan n natural sonni topishimiz kerak:

$$\frac{1}{(n+1)^3} < 0{,}01 \ \rightarrow 100 < (n+1)^3 \ \rightarrow \sqrt[3]{100} - 1 < n = 4.$$

Demak s_4 qismiy yig‘indini hisoblaymiz:

$$s_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} \approx 0{,}896412. \blacktriangleleft$$

11.7. Funksional qatorlar

Biror D to‘plamda aniqlangan funksiyalarning

$$u_1(x), u_2(x), \ldots, u_k(x), \ldots$$

cheksiz ketma-ketligi berilgan bo‘lsin.

11.9-Ta’rif. $u_1(x), u_2(x), \ldots, u_k(x), \ldots$ funksiyalar ketma-ketligi aniqlagan

$$u_1(x) + u_2(x) + \cdots + u_k(x) + \cdots = \sum_{n=1}^{\infty} u_n(x) \qquad (11.30)$$

ifodaga funksional qator deb ataymiz.

(11.30) funksional qatorda x o‘zgaruvchiga aniq bir x_0 qiymat berib yaqinlashuvchi yoki uzoqlashuvchi

$$\sum_{n=1}^{\infty} u_n(x_0)$$

sonli qatorga ega bo‘lamiz.

11.10-Ta’rif. Agar $\sum_{n=1}^{\infty} u_n(x_0)$ sonli qator yaqinlashuvchi bo‘lsa, (11.30) funksional qator x_0 nuqtada yaqinlashuvchi, agar bu sonli qator uzoqlashsa funksional qator bu nuqtada uzoqlashuvchi deyiladi.

11.11-Ta’rif. Funksional qator yaqinlashuvchi bo‘ladigan barcha nuqtalarning D to‘plamiga uning yaqinlashish sohasi deyiladi.

Funksional qatorning yaqinlashish sohasida uning yig‘indisi x o‘zgaruvchining biror funksiyasi bo‘ladi: $s = s(x)$.

Masalan,

$$1 + x + x^2 + \cdots + x^n + \cdots$$

qator $(-1;1)$ oraliqda yaqinlashadi va bu oraliqda uning yig‘indisi $\frac{1}{1-x}$ ga teng bo‘ladi:

$$1+x+x^2+\cdots+x^n+\cdots=\frac{1}{1-x}, x\in(-1;1).$$

11.26-Misol. $\sum\limits_{n=1}^{\infty}\frac{\operatorname{tg}^n x}{n}$ funksional qator yaqinlashish sohasini toping.

►Bu funksional qatorning absolyut yaqinlashish sohasi $|\operatorname{tg} x|<1$ tengsizlik bilan aniqlanadi. $\operatorname{tg} x=1$ bo‘lsa, uzoqlashuvchi garmonik qator, $\operatorname{tg} x=-1$ bo‘lsa, shartli yaqinlashuvchi (11.21-Misol) qator hosil bo‘ladi. $|\operatorname{tg} x|<1$ tengsilikning yechimlari $-\frac{\pi}{4}+\pi n<x<\frac{\pi}{4}+\pi n$, $n\in\mathbb{Z}$ to‘plamdan iborat bo‘ladi. Shunday qilib, berilgan funksional qator $-\frac{\pi}{4}+\pi n<x<\frac{\pi}{4}+\pi n$ oraliqlarda absolyut yaqinlashadi, $\left\{-\frac{\pi}{4}+\pi n\right\}$ nuqtalarda shartli yaqinlashadi va qolgan nuqtalarda uzoqlashadi.◄

11.27-Misol. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^2}$ funksional qator yaqinlashish sohasini toping.

►Ixtiyoriy $x\in\mathbb{R}$ nuqtalarda $\left|\frac{\sin nx}{n^2}\right|\le\frac{1}{n^2}$ tengsizliklar o‘rinli, $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ qator esa yaqinlashuvchi (11.13-Misol). Shuning uchun 11.6-Teoremaga ko‘ra barcha $x\in\mathbb{R}$ nuqtalarda berilgan qator absolyut yaqinlashadi. Shunday qilib berilgan qatorning absolyut yaqinlashish sohasi $\mathbb{R}=(-\infty;+\infty)$ son o‘qidan iborat ekan.◄

11.27-Misol. $\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x+1}\right)^n, x\in\mathbb{R}, x\neq-1$ funksional qator yaqinlashish sohasini toping.

►Bu qator yaqinlashishini Dalamber alomatiga ko‘ra tekshiramiz:

$$\lim_{n\to+\infty}\frac{|u_{n+1}(x)|}{|u_n(x)|}=\lim_{n\to+\infty}\frac{n}{n+1}\left|\frac{x-1}{x+1}\right|=\left|\frac{x-1}{x+1}\right|.$$

Berilgan qator yaqinlashishi uchun o‘ng tomondagi ifoda birdan kichik bo‘lishi kerak: $|(x-1)/(x+1)|<1$. Bu tengsizlikni yechsak $x>0$ natijani olamiz. Xuddi shu singari, $|(x-1)/(x+1)|>1$ tengsizlikni yechsak, $x<0$ bo‘ladi. Shunday qilib, berilgan funksional qator $x>0$ bo‘lganda absolyut yaqinlashadi, $x < 0$ bo‘lganda uzoqlashadi va $x = 0$ nuqtada $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{n}$ shartli yaqinlashuvchi qatorga ega bo‘lamiz.◀

$s_n(x)$ orqali (11.30) funksional qatorning $n-$ qismiy yig‘indisini belgilaymiz:

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x).$$

Sonli qatorlardagi singari (11.30) qatorning dastlabki n ta hadini tashlab yuborishdan hosil bo‘lgan

$$u_{n+1}(x) + u_{n+2}(x) + \cdots \qquad (11.31)$$

qator (11.30) funksional qatorning $n-$qoldiq qatori deb ataladi, uning yig‘indisini esa $r_n(x)$ orqali belgilaymiz:

$$r_n = u_{n+1}(x) + u_{n+2}(x) + \cdots$$

va funksional qatorning yaqinlashish sohasida u yaqinlashuvchi bo‘ladi.

Funksional qatorning yaqinlashish sohasidan olingan ixtiyoriy x nuqtada

$$s(x) = s_n(x) + r_n(x)$$

tenglik o‘rinli, bundan tashqari bu nuqtalarda

$$\lim_{n\to+\infty} s_n(x) = s(x)$$

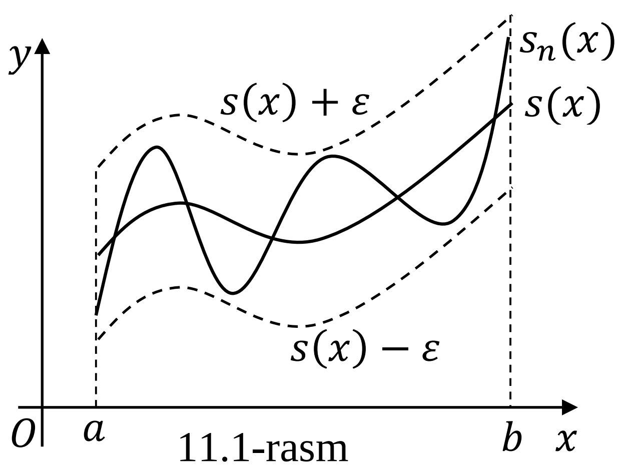
bo‘lganligi uchun

$$\lim_{n\to+\infty} r_n(x) = \lim_{n\to+\infty}[s(x) - s_n(x)] = 0$$

bo‘ladi.

$\sum\limits_{n=1}^{\infty} u_n(x)$ (U) funksional qator biror D sohada $s(x)$ funksiyaga yaqinlashsin va $X \subseteq D$ bo‘lsin.

11.12-Ta’rif. Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday N natural son topilib, $n > N$ tengsizlikni qanoatlantiruvchi barcha n sonlarda va barcha $x \in X$ nuqtalarida $|s(x) - s_n(x)| < \varepsilon$ tengsizlik o‘rinli bo‘lsa, (U) funksional qator X to‘plamda $s(x)$ funksiyaga tekis yaqinlashuvchi deyiladi.



11.1-rasm

To‘plamda tekis yaqinlashish tushunchasining geometrik talqinini beramiz. $s_n(x)$ qismiy yig‘indilar ketma-ketligi $[a; b]$ kesmada $s(x)$ funksiyaga tekis yaqinlashsin. $y = s(x) - \varepsilon$ va $y = s(x) + \varepsilon$ funksiyalarning grafigini chizamiz (11.1-rasm). U holda $\{s_n(x)\}$ qismiy yig'indilar ketma-ketligining $s(x)$ funksiyaga tekis yaqinlashgan-ligi uchun yetarlicha katta n larda va barcha $x \in [a;b]$ nuqtalarda

$$s(x) - \varepsilon < s_n(x) < s(x) + \varepsilon$$

tengsizliklar o'rinli bo'ladi. Bu esa qandaydir $N(\varepsilon)$ tartib raqamidan boshlab $s_n(x)$ funksiyalar grafigi $y = s(x) - \varepsilon$ va $y = s(x) + \varepsilon$ funksiyalar grafiklari bilan chegaralangan $\varepsilon -$yo'lakada yotishini anglatadi (11.1-rasm).

11.13-Ta'rif. Agar barcha $x \in X$ nuqtalarda $\sum\limits_{n=1}^{\infty} u_n(x)$ (U) funksional qatorning hadlari absolyut qiymati bo'yicha yaqinlashuvchi musbat hadli $\sum\limits_{n=1}^{\infty} a_n$ (A) qatorning mos hadlaridan katta bo'lmasa, ya'ni ixtiyoriy $x \in X$ nuqtalarda

$$|u_n(x)| \leq a_n, n \in \mathbb{N} \tag{11.32}$$

tengsizliklar o'rinli bo'lsa, berilgan (U) funksional qator X to'plamda kuchaytirilgan funksional qator deb ataladi.

11.18-Teorema (Veyyershtass alomati). X to'plamda kuchaytirilgan funksional qator bu to'plamda tekis va absolyut yaqinlashuvchi bo'ladi.

► X to'plamning barcha nuqtalarida (U), (A) qatorlar hadlari uchun (11.32) tengsizliklar o'rinli bo'lsin. U holda 11.6-Teoremaga ko'ra ixtiyoriy $x \in X$ nuqtada $\sum\limits_{n=1}^{\infty} |u_n(x)|$ qator yaqinlashuvchi bo'ladi, ya'ni (U) qator absolyut yaqinlashuvchi.

Endi (U) qatorning tekis yaqinlashishini ko'rsatamiz.

$$\sum_{n=1}^{\infty} u_n(x) = s(x) \text{ va } \sum_{n=1}^{\infty} a_n = a$$

bo'lsin. U holda ixtiyoriy $x \in X$ nuqtada

$$|s(x) - s_n(x)| = |r_n(x)| = |u_{n+1}(x) + u_{n+2}(x) + \cdots| \leq$$
$$\leq |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots \leq a_{n+1} + a_{n+2} + \cdots = \sigma_n$$

tengsizliklar o'rinli bo'ladi, bu yerda σ_n yuqorida keltirilgan (A) qatorning qoldiq qatori. 11.1-Teoremaga ko'ra σ_n qoldiq qator n cheksiz o'sganda nolga intiladi. Shuning uchun ketma-ketlikning ta'rifiga ko'ra, ixtiyoriy $\varepsilon > 0$ son uchun shunday $N(\varepsilon)$ tartib raqami topiladidki, barcha $n > N(\varepsilon)$ larda $\sigma_n < \varepsilon$

tengsizlik bajariladi. Yuqoridagi tengsizlikdan esa xuddi shunday n sonlari uchun

$$|s(x) - s_n(x)| < \varepsilon$$

tengsizlik ixtiyoriy $x \in X$ nuqtada bajarilishi kelib chiqadi. Bu esa 11.12-Ta'rifdan (U) qatorning tekis yaqinlashishini anglatadi.◄

11.28-Misol. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+x^2}$ funksional qator ℝ son o'qida tekis yaqinlashuvchi ekanlgini isbotlang.

►Ixtiyoriy $x \in \mathbb{R}$ nuqtalarda $\frac{1}{n^2+x^2} < \frac{1}{n^2}$ tengsizlik o'rinli va $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ sonli qator yaqinlashuvchi (11.13-Misol). U holda Veyyershtass alomatiga ko'ra berilgan qator tekis yaqinlashuvchi.◄

11.29-Misol. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+x^2}$ funksional qator yaqinlashishini tekshiring.

►Berilgan qator ishoralari navbatlashuvchi qator va u Leybnis teoremasiga ko'ra butun son o'qida yaqinlashuvchi. Bu yaqinlashish shartli, chunki qator hadlarining absolyut qiymatlaridan tuzilgan $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}+x^2}$ qator butun son o'qida uzoqlashuvchi. Shuning uchun funksional qatorning tekis yaqinlashuvchiligini tekshirishga Veyyershtass alomatini qo'llab bo'lmaydi. Bu holda biz ishoralari navbatlashuvchi qatorning qoldig'i uchun 11.2-Natijadan foydalanamiz:

$$|r_n(x)| = \left|\sum_{k=n+1}^{\infty} \frac{(-1)^n}{\sqrt{n}+x^2}\right| \le \left|\frac{(-1)^n}{\sqrt{n+1}+x^2}\right| = \frac{1}{\sqrt{n+1}+x^2} \le \frac{1}{\sqrt{n+1}}.$$

Endi tekis yaqinlashish ta'rifiga ko'ra $|s(x) - s_n(x)| = |r_n(x)| < \varepsilon$ tengsizlik o'rinli bo'ladigan n uchun $N(\varepsilon)$ chegarani topishimiz kerak. Uni biz yuqorida hosil qilingan $|r_n(x)| \le 1/\sqrt{n+1} < \varepsilon$ tengsizlikdan aniqlaymiz. Bu tengsizlikdan ixtiyoriy $\varepsilon > 0$ soni uchun shunday $N(\varepsilon) = (\varepsilon^{-2} - 1)$ topilib, barcha $n > N(\varepsilon)$ uchun $|s(x) - s_n(x)| < \varepsilon$ tengsizlikning bajarilishi kelib chiqadi. Bu esa berilgan qatorning butun son o'qida tekis yaqinlashishini anglatadi.◄

11.19-Teorema. Agar $\sum\limits_{n=1}^{\infty} u_n(x)$ (U) va $\sum\limits_{n=1}^{\infty} v_n(x)$ (V) funksional qatorlar biror X to'plamda tekis yaqinlashuvchi bo'lsa, u holda ularning ixtiyoriy

$$\sum_{n=1}^{\infty}(\alpha u_n(x)+\beta v_n(x)), \alpha, \beta \in \mathbb{R} \qquad (11.33)$$

chiziqli kombinatsiyasi ham ana shu X to‘plamda tekis yaqinlashuvchi bo‘ladi.

►(U) va (V) qatorlarning yig‘indisi va $n-$qismiy yig‘indisi mos ravishda $s(x), s_n(x)$ va $t(x), t_n(x)$ bo‘lsin. X to‘plamdan olingan har bir $x \in X$ nuqtada (11.33) sonli qatorga aylanadi va u 11.3 va 11.4-Teoremalarga ko‘ra yaqinlashuvchi bo‘lib, uning yig‘indisi $\alpha s(x)+\beta t(x)$ qiymatga teng. (U) va (V) qatorlar tekis yaqinlashuvchi ekanligidan ixtiyoriy $\varepsilon/(|\alpha|+|\beta|)$ son uchun shunday $N(\varepsilon)$ topiladiki, ixtiyoriy $x \in X$ nuqtalarda va $n > N(\varepsilon)$ natural sonlarda

$$\left|\left(\alpha s(x)+\beta t(x)\right)-\left(\alpha s_n(x)+\beta t_n(x)\right)\right| =$$
$$= |\alpha(s(x)-s_n(x))+\beta(t(x)-t_n(x))| \leq$$
$$\leq |\alpha||s(x)-s_n(x)|+|\beta||t(x)-t_n(x)| < \frac{|\alpha|\varepsilon}{|\alpha|+|\beta|}+\frac{|\beta|\varepsilon}{|\alpha|+|\beta|} = \varepsilon$$

tengsizlik o‘rinli bo‘ladi. Bu esa (11.33) qatorning tekis yaqinlashuvchi ekanligini anglatadi.◄

11.20-Teorema. Agar X to‘plamda tekis yaqinlashuvchi $\sum_{n=1}^{\infty} u_n(x)$ (U) qatorni shu to‘plamda chegaralangan $v(x)$ funksiyaga ko‘paytirilsa, hosil qilingan qator X to‘plamda tekis yaqinlashadi.

► $v(x)$ funksiya X to‘plamda chegaralangan, ya’ni $|v(x)| < L, x \in X$, hamda (U) qatorning yig‘indisi va $-$qismiy yig‘indisi $s(x)$ va $s_n(x)$ bo‘lsin. U holda (U) qatorning X to‘plamda tekiz yaqinlashuvchiligidan uning bu to‘plamda nuqtaviy yaqinlashishi kelib chiqadi va $x \in X$ o‘zgaruvchining aniq bir qiymatida sonli qator hosil bo‘ladi. Sonli qatorlarning ma’lum xossalaridan

$$\sum_{n=1}^{\infty} v(x)\cdot u_n(x) = v(x)\cdot\sum_{n=1}^{\infty} u_n(x) = v(x)s(x)$$

tenglikni hosil qilamiz. Shuning uchun

$$\left|v(x)s(x)-\sum_{k=1}^{n} v(x)\cdot u_k(x)\right| = \left|v(x)s(x)-v(x)\cdot\sum_{k=1}^{n} u_k(x)\right| =$$
$$= |v(x)||s(x)-s_n(x)| \leq L\varepsilon$$

tengsizlik o‘rinli bo‘ladi. Bu esa $\sum_{n=1}^{\infty} v(x)\cdot u_n(x)$ qatorning tekis yaqinlashishini anglatadi.◄

11.21-Teorema (tekis yaqinlashuvchi funksional qator yig‘indisining uzluksizligi). Agar X to‘plamda tekis yaqinlashuvchi $\sum_{n=1}^{\infty} u_n(x)$ (U) qatorning $u_1(x), u_2(x), \ldots$ hadlari shu to‘plamda uzluksiz bo‘lsa, bu qatorning $s(x)$ yig‘indisi ham X to‘plamda uzluksiz bo‘ladi.

► Qatorning $s(x)$ yig‘indisini

$$s(x) = s_n(x) + r_n(x)$$

ko‘rinishda yozib olamiz. (U) qatorning tekis yaqinlashuvchiligidan n −tartib raqamini barcha $x \in X$ uchun $|r_n(x)| < \varepsilon/4$ bo‘ladigan qilib tanlash mumkin. U holda ixtiyoriy $x, x' = x + \Delta x \in X$ nuqtalar uchun

$$|r_n(x) - r_n(x')| < |r_n(x)| + |r_n(x')| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

tengsizlik o‘rinli.

$s_n(x)$ qismiy yig‘indi X to‘plamda uzluksiz bo‘lgan chekli sondagi funksiyalarning yig‘indisidan iborat, shuning uchun uning o‘zi ham bu to‘plamda uzluksiz bo‘ladi. Shu tufayli ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topiladiki, $|\Delta x| < \delta$ tengsizlikni qanoatlantiruvchi barcha Δx qiymatlarda $|s_n(x + \Delta x) - s_n(x)| < \varepsilon/2$ tengsizlik o‘rinli bo‘ladi. Bundan esa funksiyaning uzluksizligini ifodalovchi quyidagi

$$|s(x + \Delta x) - s(x)| = |s_n(x + \Delta x) + r_n(x + \Delta x) - s_n(x) + r_n(x)| \leq$$
$$\leq |s_n(x + \Delta x) - s_n(x)| + |r_n(x + \Delta x) - r_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

tengsizlikni hosil qilamiz.◄

11.22-Teorema (tekis yaqinlashuvchi funksional qatorni hadma-had integrallash). Agar $[a; b]$ kesmada $s(x)$ funksiyaga tekis yaqinlashuvchi $\sum_{n=1}^{\infty} u_n(x)$ (U) funksional qatorning $u_1(x), u_2(x), \ldots$ hadlari uzluksiz bo‘lsa, (U) qatorni ixtiyoriy $[x_0; x] \subseteq [a; b]$ kesma bo‘yicha hadma-had integrallash mumkin, ya’ni ixtiyoriy $x, x_0 \in [a; b]$ uchun

$$\int_{x_0}^{x} s(t)dt \equiv \int_{x_0}^{x} \left[\sum_{n=1}^{\infty} u_n(t)\right] dt = \sum_{n=1}^{\infty} \left[\int_{x_0}^{x} u_n(t)dt\right]$$

tenglik o‘rinli bo‘ladi. $x_0 \in [a; b]$ nuqta qanday bo‘lishidan qat’iy nazar hosil bo‘lgan qator $[a; b]$ kesmada x bo‘yicha tekis yaqinlashuvchi bo‘ladi.

► $u_n(x)$ funksiyalar uzluksiz va (U) qator $[a; b]$ kesmada tekis yaqinlashuvchi bo‘lganligi uchun 11.21-Teoremaga ko‘ra qatorning $s(x)$ yig‘indisi ham uzluksiz va demak bu kesmada integrallanuvchi bo‘ladi. $s_n(x)$

(U) qatorning $n-$ qismiy yig‘indisi bo‘lsin. U holda qatorning tekis yaqinlashuvchi ekanligidan ixtiyoriy $\varepsilon > 0$ son uchun shunday $N(\varepsilon)$ natural son topilib, barcha $n > N(\varepsilon)$ sonlarda va barcha $x \in [a; b]$ nuqtalarda

$$|s(x) - s_n(x)| < \frac{\varepsilon}{b-a}$$

tengsizlining bajarilishi kelib chiqadi. O‘z navbatida bu tengsizlikni qo‘llab ixtiyoriy $n > N(\varepsilon)$ uchun

$$\left|\int_{x_0}^{x} s(t)dt - \int_{x_0}^{x} s_n(t)dt\right| \leq \int_{x_0}^{x} |s(t) - s_n(t)|dt < \int_{x_0}^{x} \frac{\varepsilon}{b-a}dt =$$

$$= \frac{\varepsilon}{b-a}|x - x_0| < \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

tengsizlikning bajarilishiga kelamiz. Bu esa

$$\int_{x_0}^{x} s(t)dt = \lim_{n\to+\infty}\int_{x_0}^{x} s_n(t)dt = \lim_{n\to+\infty}\int_{x_0}^{x}\left[\sum_{k=1}^{n} u_k(t)\right]dt =$$

$$= \lim_{n\to+\infty}\sum_{k=1}^{n}\int_{x_0}^{x} u_k(t)dt$$

yoki

$$\int_{x_0}^{x} s(t)dt = \sum_{n=1}^{\infty}\int_{x_0}^{x} u_n(t)dt$$

ekanligidan dalolat beradi.◄

11.30-Misol. $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(2n+1)}$ qator yig‘indisini toping.

► $\sum\limits_{n=0}^{\infty}(-1)^n x^{2n}$ funksional qator $[-\alpha; \alpha], 0 < \alpha < 1$ kesmada tekis yaqinlashadi va uning yig‘indisi $\frac{1}{1+x^2}$ qiymatga teng (geomatrik progressiya). Bu qatorni hadma-had integrallashdan hosil bo‘lgan $\sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ qator ham $[-\alpha; \alpha]$ kesmada tekis yaqinlashadi va uning yigindisi uchun

$$\int_{0}^{x}\frac{1}{1+t^2}dt = \operatorname{arctg} t\Big|_0^x = \operatorname{arctg} x$$

tenglik o‘rinli. Yig‘indisini hisoblash talab qilingan qator esa $\sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ qatorning $x=1/2$ nuqtadagi qiymatiga teng. Shuning uchun $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(2n+1)}=\operatorname{arctg}\left(\frac{1}{2}\right)$.◄

11.31-Misol. $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+x^2}$ funksional qatorni hadma-had integrallash mumkinligini isbotlang.

►Funksional qatorning har bir hadi uzluksiz funksiyadan iborat va qator kuchaytirilgan, chunki ixtiyoriy x uchun $\frac{1}{n^2+x^2}<\frac{1}{n^2}$ va $\sum\limits_{n=0}^{\infty}\frac{1}{n^2}$ sonli qator yaqinlashuvchi. Veyyershtrass alomatiga ko‘ra berilgan funksional qator $\mathbb{R}$ son o‘qida tekis yaqinlashadi va 11.22-Teoremaga ko‘ra uni hadma-had integrallash mumkin:

$$\int\limits_0^x\left[\sum_{n=1}^{\infty}\frac{1}{n^2+t^2}\right]dt=\sum_{n=1}^{\infty}\int\limits_0^x\frac{1}{n^2+t^2}dt=\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{arctg}\frac{t}{n}\Big|_0^x=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{arctg}\frac{x}{n}.\ ◄$$

11.23-Teorema (funksional qatorni hadma-had differensiallash). $[a;b]$ kesmada yaqinlashuvchi $\sum\limits_{n=1}^{\infty}u_n(x)$ (U) funksional qatorning $u_1(x),u_2(x),\ldots$ hadlari uzluksiz $u_1'(x),u_2'(x),\ldots$ hosilalarga ega va bu hosilalardan tuzilgan $\sum\limits_{n=1}^{\infty}u_n'(x)$ (U′) qator shu kesmada tekis yaqinlashsin. U holda ixtiyoriy $x\in[a;b]$ nuqtada

$$\left[\sum_{n=1}^{\infty}u_n(x)\right]'=\sum_{n=1}^{\infty}u_n'(x)$$

tenglik o‘rinli bo‘ladi, ya’ni berilgan funksional qatorni hadma-had differensiallash mumkin.

►(U) va (U′) qatorlarning yig‘indilari mos ravishda $s(x)$ va $v(x)$ funksiyalar bo‘lsin. U holda 11.22-Teoremaga ko‘ra ixtiyoriy $x,x_0\in[a;b]$ nuqtalar uchun

$$\int\limits_{x_0}^{x}v(t)dt=\int\limits_{x_0}^{x}\left[\sum_{n=1}^{\infty}u_n'(t)\right]dt=\sum_{n=1}^{\infty}\int\limits_{x_0}^{x}u_n'(t)dt=$$

$$= \sum_{n=1}^{\infty}[u_n(x) - u_n(x_0)] = \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(x_0) = s(x) - s(x_0)$$

tenglik o‘rinli bo‘ladi. $v(x)$ funksiya 11.21-Teoremaga ko‘ra uzluksiz. Shuning uchun

$$\int_{x_0}^{x} v(t)dt = s(x) - s(x_0)$$

tenglikni differensiallasak

$$\left[\int_{x_0}^{x} v(t)dt\right]' = s'(x)$$

ya’ni $v(x) = s'(x)$ va demak

$$\left[\sum_{n=1}^{\infty} u_n(x)\right]' = \sum_{n=1}^{\infty} u'_n(x)$$

tenglikni hosil qilamiz.◄

11.32-Misol. $\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \cdots + \frac{n}{3^n} + \cdots$ qator yigindisini hisoblang.

► $\sum_{n=0}^{\infty} e^{nx}$ funksional qatorni qaraymiz. Bu qator ixtiyoriy $x \in \mathbb{R}$ nuqtada musbat hadli qator bo‘ladi va u $(-\infty; 0)$ oraliqda yaqinlashadi. Ixtiyoriy $x \in (-\infty; 0)$ nuqtada $\sum_{n=0}^{\infty} e^{nx} = \frac{1}{1-e^x}$ (cheksiz kamayuvchi geometrik progressiya). Uning hadlarining hosilalaridan $\sum_{n=0}^{\infty} ne^{nx}$ qatorni tuzamiz va u $(-\infty; \alpha)$, $\alpha < 0$ oraliqda tekis yaqinlashishini ko‘rsatamiz. Dastlab $\sum_{n=0}^{\infty} ne^{n\alpha}$ sonli qatorni qaraylik. U yaqinlashuvchi, chunki Dalamber alomatiga ko‘ra

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{(n+1)e^{(n+1)\alpha}}{ne^{n\alpha}} = \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)e^{\alpha} = e^{\alpha} < 1.$$

$x < \alpha$ bo‘lganda $ne^{nx} < ne^{n\alpha}$ va $\sum_{n=0}^{\infty} ne^{n\alpha}$ sonli qator yaqinlashuvchi, shuning uchun $\sum_{n=0}^{\infty} ne^{nx}$ funksional qator kuchaytirilgan. U holda Veyyershtrass alomatiga ko‘ra bu qator $(-\infty; \alpha), \alpha < 0$ oraliqda tekis yaqinlashadi. Shuning uchun $\sum\limits_{n=0}^{\infty} e^{nx}$ funksional qatorga 11.23-Teoremani qo‘llash mumkin:

$$\left(\sum_{n=0}^{\infty} e^{nx}\right)' = \sum_{n=0}^{\infty} ne^{nx},$$

yoki

$$\left(\sum_{n=0}^{\infty} e^{nx}\right)' = \left(\frac{1}{1-e^x}\right)' = \frac{e^x}{1-e^x} = \sum_{n=0}^{\infty} ne^{nx}.$$

Shunday qilib,

$$\sum_{n=0}^{\infty} ne^{nx} = \frac{e^x}{(1-e^x)^2}.$$

Agar bu tenglikda $x = -\ln 3 = \ln\frac{1}{3}$ deb olsak, chap tomonda yig‘indisi hisoblanishi talab qilingan $\sum\limits_{n=0}^{\infty} \frac{n}{3^n}$ qatorni hosil qilamiz. Shuning uchun

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \frac{e^{\ln\frac{1}{3}}}{\left(1-e^{\ln\frac{1}{3}}\right)^2} = \frac{3}{4}$$

sonli qator yig‘indisi bo‘ladi.◄

11.8. Darajali qatorlar

11.14-Ta’rif. Darajali qator deb

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n \qquad (11.34)$$

yoki

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots =$$
$$= \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad (11.35)$$

ko‘rinishdagi funksional qatorlarni datajali qator deb ataymiz, bu yerda $a_0, a_1, a_2, \dots, a_n, \dots$ haqiqiy (yoki kompleks) sonlar bo‘lib, ular (11.34) va (11.35) darajali qatorlarning koeffisiyentlari deb ataladi, $x \in \mathbb{R} -$ haqiqiy o‘zgaruvchi.

(11.34) qator $x = 0$ nuqtada, (11.35) qator esa $x = x_0$ nuqtada yaqinlashadi va ularning yig‘indisi a_0 qiymatga teng. (11.35) qatorda $x - x_0$ ifodani x bilan almashtirilsa (11.34) qator hosil bo‘ladi. Shuning uchun biz

(11.34) ko‘rinishdagi darajali qatorlarni qarash bilan chegaralanamiz va ularga nisbatan olinadigan barcha natijalar (11.35) qator uchun ham o‘rinli. Bundan tashqari ayrim formulalarni (11.35) qator uchun ham alohida keltiramiz.

11.24-Teorema (Abel). Agar $\sum_{n=0}^{\infty} a_n x^n$ (A) qator biror $x = x_1 \neq 0$ nuqtada yaqinlashsa, u holda u $|x| < |x_1|$ tengsizlikni qanoatlantiruvchi barcha x nuqtalarda absolyut yaqinlashadi; agar (A) qator biror $x = x_2$ nuqtada uzoqlashsa, u holda u $|x| > |x_1|$ tengsizlikni qanoatlantiruvchi barcha x nuqtalarda uzoqlashadi.

► (A) qator $x = x_1 \neq 0$ nuqtada yaqinlashuvchi bo‘lsin. U holda qator yaqinlashishining zaruriy shartiga ko‘ra $\lim_{n\to+\infty} a_n x_0^n = 0$ bo‘ladi. Bundan esa $a_n x_0^n$ miqdorning chegaralanganligi, ya’ni shunday $M > 0$ son topilib, barcha n uchun $|a_n x_0^n| \leq M$ tengsizlikning bajarilishi kelib chiqadi.

$|x| < |x_0|$ bo‘lsin, u holda $q = \left|\frac{x}{x_0}\right| < 1$ bo‘ladi. Demak,

$$|a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x^n}{x_0^n}\right| \leq M \cdot q^n, n = 0,1,2, \ldots,$$

ya’ni (A) qator barcha hadlarining absolyut qiymatlari cheksiz kamayuvchi geometrik progressiyaning mos hadlaridan katta emas. Shuning uchun (A) qator 11.6-Teoremaga ko‘ra absolyut yaqinlashuvchi.

(A) qator $x = x_2$ nuqtada uzoqlashuvchi bo‘lsin. Faraz qilaylik qator biror x nuqtada yaqinlashuvchi bo‘lsin. U holda yuqorida isbotlaganimizga ko‘ra qator $x = x_2$ nuqtada yaqinlashuvchi bo‘lishi kerak, chunki $|x_2| < |x|$. Bu esa teorema sharti bo‘yicha qatorning $x = x_2$ nuqtada uzoqlashuvchi ekanligiga ziddir. ◄

Abel teoremasidan shunday muhim xulosani chiqarish mumkin: har qanday darajali qator uchun shunday $R \geq 0$ soni mavjudki, bunda

a) barcha $|x| < R$ nuqtalarda (A) qator absolyut yaqinlashadi;
b) barcha $|x| > R$ nuqtalarda (A) qator uzoqlashadi;
c) $x = \pm R$ nuqtalarda mos $\sum_{n=0}^{\infty} a_n R^n$ va $\sum_{n=0}^{\infty} a_n (-R)^n$ sonli qatorlar yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin.

$R = \infty$ bo‘lsa, (A) qator son o‘qining ixtiyoriy $x \in R$ nuqtasida yaqinlashadi. $R = 0$ bo‘lsa, qator faqat $x = 0$ nuqtadagina yaqinlashadi.

11.15-Ta’rif. $\sum\limits_{n=0}^{\infty} a_n x^n$ (A) darajali qatorning yaqinlashish oralig‘i deb, shunday $(-R;R), R>0$ oraliqqa aytiladiki, bunda barcha $x\in(-R;R)$ nuqtalarda qator absolyut yaqinlashadi, $|x|>R$ tengsizlikni qanoatlantiruvchi x nuqtalarda esa qator uzoqlashadi. R soni darajali qatorning yaqinlashish radiusi deb ataladi.

Darajali qator yaqinlashish radiusini hisoblash usullarini ko‘rsatamiz.

$\sum\limits_{n=0}^{\infty} |a_n||x^n|$ darajali qatorning yaqinlashish radiusi $\sum\limits_{n=0}^{\infty} a_n x^n$ darajali qatorning radiusi bilan bir xil bo‘ladi. Shuning uchun musbat hadli qatorlarning xossalaridan foydalanish mumkin.

$\sum\limits_{n=0}^{\infty} |a_n||x^n|$ musbat hadli qator uchun Dalamber alomatini qo‘llaymiz:

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Agar so‘ngi limit mavjud bo‘lsa, qator yaqinlashishi uchun

$$|x|\cdot\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$

yoki

$$|x|<\frac{1}{\lim\limits_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

tengsizlik o‘rinli bo‘lishi kerak. Shuning uchun

$$R=\frac{1}{\lim\limits_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|}=\lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (11.36)$$

bo‘ladi.

Xuddi shu singari, agar $\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}$ limit mavjud bo‘lsa, Koshi alomatini qo‘lab, darajali qator yaqinlashish radiusi uchun

$$R=\frac{1}{\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}} \qquad (11.37)$$

tenglikni hosil qilamiz.

$\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ ko‘rinishdagi qatorning ham yaqinlashish radiusi (11.36) yoki (11.37) tengliklar bilan aniqlanadi. Faqat bu holda yaqinlashish oralig‘i $(x_0-R;x_0+R)$ ko‘rinishda bo‘ladi.

11.33-Misol. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}(x-1)^n$ qatorning yaqinlashish sohasini toping.

►Bu yerda $x_0 = 1$ va $a_n = 1/n^2$, $a_{n+1} = 1/(n+1)^2$, u holda

$$R = \lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty}\frac{1/n^2}{1/(n+1)^2} = \lim_{n\to+\infty}\frac{(n+1)^2}{n^2} =$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^2 = 1.$$

Shuning uchun berilgan qator $(1-1; 1+1) = (0; 2)$ oraliqda yaqinlashadi.

Endi oraliqning chetki nuqtalarida qator yaqinlashishini tekshiramiz. $x = 0$ bo‘lsa, ishoralari navbatlashuvchi $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ qator hosil bo‘ladi va absolyut yaqinlashuvchi, chunki hadlarining absolyut qiymatlaridan tuzilgan $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ qator yaqinlashuvchi (11.13-Misol). $x = 2$ bo‘lsa, yana yaqinlashuvchi $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ musbat hadli qator hosil bo‘ladi.

Shunday qilib, berilgan darajali qatorning yaqinlashish sohasi $[0; 2]$ kesmadan iborat ekan.◄

11.34-Misol. $\sum\limits_{n=1}^{\infty} \frac{x^{2n}}{3^n}$ (X) qatorning yaqinlashish sohasini toping.

►Maskur qatorda x o‘zgaruvchining toq darajalari yo‘q. Shuning uchun $y = x^2$ deb, qator ko‘rinishini o‘zgartiramiz. U holda $\sum\limits_{n=1}^{\infty} \frac{y^n}{3^n}$ (Y) qator hosil bo‘ladi va bu yerda $a_n = 1/3^n$. Hosil bo‘lgan (Y) qatorning yaqinlashish yaqinlashish radiusini aniqlash uchun (11.37) formuladan foydalanamiz:

$$R = \frac{1}{\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}} = \lim_{n\to+\infty}\sqrt[n]{3^n} = 3.$$

Demak (Y) qator $(-3; 3)$ oraliqda yaqinlashar ekan. $y = x^2$ tenglikdan $x = \pm\sqrt{y}$ va bundan (X) qator uchun $(-\sqrt{3};\ \sqrt{3})$ yaqinlashish oralig‘ini topamiz. Oraliqning chetki nuqtalarida qator yaqinlashishini tekshiramiz. $x = \pm\sqrt{3}$ nuqtalarda $\sum\limits_{n=1}^{\infty} \frac{(\pm\sqrt{3})^{2n}}{3^n} = 1 + 1 + \cdots + 1 + \cdots$ uzoqlashuvchi qator hosil bo‘ladi. Shuning uchun berilgan darajali qatorning yaqinlashish sohasi $(-\sqrt{3};\ \sqrt{3})$ oraliqdan iborat bo‘ladi.◄

11.25-Teorema. $\sum\limits_{n=0}^{\infty} a_n x^n$ (A) darajali qatorning $s(x)$ yig‘indisi uning $(-R;R), R>0$ yaqinlashish oralig‘ida uzluksiz funksiya bo‘ladi.

► Ixtiyoriy $[-r;r] \subset (-R;R)$ kesmani olamiz. r nuqta (A) darajali qatorning yaqinlashish oralig‘iga tegishli bo‘lganligi sababli $\sum\limits_{n=0}^{\infty} |a_n r^n|$ musbat hadli qator yaqinlashuvchi bo‘ladi. Bundan tashqari ixtiyoriy $x \in [-r;r]$ nuqtada $|a_n x^n| \le |a_n r^n|$ tengsizlik o‘rinli, ya’ni (A) qator $[-r;r]$ kesmada kuchaytirilgan. (A) qatorning $a_n x^n$ hadlari bu kesmada uzluksiz bo‘lganligi uchun 11.21-Teoremaga ko‘ra qatorning $s(x)$ yig‘indisi $[-r;r]$ kesmada uzluksiz bo‘ladi. O‘z navbatida $[-r;r] \subset (-R;R)$ kesma ixtiyoriy ekanligidan, $s(x)$ yig‘indi $(-R;R)$ oraliqning ixtiyoriy nuqtasida uzluksizligi kelib chiqadi.◄

11.26-Teorema. Yig‘indisi $s(x)$ bo‘lgan $\sum\limits_{n=0}^{\infty} a_n x^n$ (A) darajali qatorni uning $(-R;R), R>0$ yaqinlashish oralig‘ida hadma-had integrallash mumkin va hadma-had integrallashdan hosil bo‘lgan qatorning yaqinlashish radiusi ham R soniga teng. Bunda ixtiyoriy $x \in (-R;R)$ nuqta uchun

$$\int_0^x s(t)dt = \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \left(\int_0^x a_n t^n dt\right) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

tenglik o‘rinli bo‘ladi.

►Bu yerda ham ixtiyoriy $[-r;r] \subset (-R;R)$ kesmani olamiz va bu kesmada (A) qator kuchaytirilgan bo‘ladi. 11.22-Teoremaga ko‘ra (A) qatorni $[-r;r]$ kesmada hadma-had integrallash mumkin va natijada teoremadagi tenglik hosil bo‘ladi.

Endi hadma-had integrallashdan hosil bo‘lgan $\sum\limits_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ qatorning yaqinlashish radiusini (11.36) formulaga ko‘ra topamiz:

$$R_1 = \lim_{n\to+\infty} \left|\frac{a_n/(n+1)}{a_{n+1}/(n+2)}\right| = \lim_{n\to+\infty} \frac{n+2}{n+1} \left|\frac{a_n}{a_{n+1}}\right| =$$

$$= \lim_{n\to+\infty} \frac{n+2}{n+1} \cdot \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = R.$$

Agar $\lim\limits_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right|$ limit mavjud bo‘lmasa, isbotni Abel teoremasining isboti singari bajarish mumkin.

$[-r;r]$ kesma ixtiyoriy ekanligidan, qatorni $(-R;R)$ yaqinlashish intervalida hadma-had integrallash mumkinligi kelib chiqadi ◄

11.27-Teorema. Yig‘indisi $s(x)$ bo‘lgan $\sum\limits_{n=0}^{\infty} a_n x^n$ (A) darajali qatorni uning $(-R;R), R>0$ yaqinlashish oralig‘ining ixtiyoriy nuqtasida hadma-had differensiallash mumkin va hadma-had differensiallashdan hosil bo‘lgan qatorning yaqinlashish radiusi ham R soniga teng. Bunda ixtiyoriy $x \in (-R;R)$ nuqta uchun

$$s'(x) \equiv \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

tenglik o‘rinli bo‘ladi.

►Teoremaning isboti 11.26-Teorema isboti singari bajariladi.◄

11.35-Misol. $\sum\limits_{n=1}^{\infty} x^{2n+1}/(2n+1)$ qatorning yaqinlashish sohasini va bu sohada uning $s(x)$ yig‘indisini toping.

►Qatorni ko‘rinishini o‘zgartiramiz:

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} = x\sum_{n=1}^{\infty} \frac{x^{2n}}{2n+1} = x\sum_{n=1}^{\infty} \frac{(x^2)^n}{2n+1}.$$

Demak $a_n = 1/(2n+1)$, $a_{n+1} = 1/(2n+3)$ va qatorning yaqinlashish radiusi

$$R = \lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{1/(2n+1)}{1/(2n+3)} = \lim_{n\to+\infty} \frac{(2n+3)}{(2n+1)} = 1$$

bo‘ladi, ya’ni berilgan darajali qator $x^2 < 1$ yoki $-1 < x < 1$ bo‘lganda yaqinlashadi. $x = \pm 1$ nuqtalarda hosil bo‘lgan sonli qatorlar uzoqlashadi.

Endi berilgan darajali qatorning yig‘indisini topamiz. Buning uchun $x^2 + x^4 + \cdots + x^n + \cdots$ darajali qatorni qaraymiz. $|x| < 1$ bo‘lganda qator cheksiz kamayuvchi geometrik progressiya sifatida $\frac{x^2}{1-x^2}$ funksiyaga tekis yaqinlashadi. Uni $[0;x] \subset (-1;1)$ kesmada hadma-had integrallasak yig‘indisini topish talab qilingan $\sum\limits_{n=1}^{\infty} x^{2n+1}/(2n+1)$ qator hosil bo‘ladi. Shuning uchun 11.26-Teoremaga ko‘ra

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \left(\int_0^x t^{2n} dt\right) = \int_0^x \left(\sum_{n=1}^{\infty} t^{2n}\right) dt =$$

$$= \int_0^x \frac{t^2}{1-t^2} dt = -\int_0^x \left(1 - \frac{1}{1-t^2}\right) dt = -x + \arcsin x.$$

Shunday qilib, berilgan $\sum_{n=1}^{\infty} x^{2n+1}/(2n+1)$ darajali qator $(-1;1)$ oraliqda $s(x) = -x + \arcsin x$ funksiyaga tekis yaqinlashar ekan.◄

11.36-Misol. $\sum_{n=1}^{\infty} x^n/n$ qatorning $s(x)$ yig'indisini toping.

►$(-1;1)$ oraliqda bu qator tekis yaqinlashadi. Shuning uchun uni bu oraliqda hadma-had differensiallash mumkin:

$$\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)' = \sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)' = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Tenglikning ikkala tomonini $[0;x], -1 < x < 1$ kesmada integrallaymiz va natijada qidirilayotgan yig'indini topamiz:

$$s(x) = \int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt = \int_0^x \frac{1}{1-t} dt = -\ln(1-x). \blacktriangleleft$$

11.28-Teorema. Yig'indisi $s(x)$ bo'lgan $\sum_{n=0}^{\infty} a_n x^n$ (A) darajali qatorni uning $(-R;R), R > 0$ yaqinlashish oralig'ining ixtiyoriy nuqtasida ixtiyoriy tartibda hadma-had differensiallash mumkin va hadma-had differensiallashdan hosil bo'lgan qatorlarning yaqinlashish radiuslari ham R soniga teng. Bunda ixtiyoriy $x \in (-R;R)$ nuqta va $k = 1,2, \ldots$ uchun

$$s^{(k)}(x) \equiv \left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=0}^{\infty} n(n-1) \ldots (n-k+1) a_n x^{n-k}$$

tenglik o'rinli bo'ladi.

►11-27-Teoremaga ko'ra (A) qatorni uning $(-R;R), R > 0$ yaqinlashish oralig'ining ixtiyoriy nuqtasida hadma-had differensiallash mumkin va hosil bo'lgan qator $(-R;R)$ oraliqda $s'(x)$ funksiyaga tekis yaqinlashadi. Shuning uchun hadma-had differensiallash amalini ixtiyoriy marta takrorlash mumkin. Bu esa teorema tasdig'ining to'g'riligini anglatadi.◄

11.37-Misol. Qator yig'indisini toping: $\sum_{n=1}^{\infty} (-1)^n \frac{n(2n-1)}{4^{n-1}}$.

►Maxraji $(-x^2)$ bo'lgan $\sum_{n=0}^{\infty} (-x^2)^n$ geometrik progressiyani qaraymiz. Bu darajali qator ixtiyoriy $(-r;r) \subset (-1;1)$ oraliqda $s(x) =$

$\sum\limits_{n=0}^{\infty}(-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ funksiyaga tekis yaqinlashadi. Shuning uchun uni ikki marta hadma-had differensiallaymiz:

$$s'(x) \equiv \left(\sum_{n=0}^{\infty}(-x^2)^n\right)' = \sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1} =$$

$$= \left(\frac{1}{1+x^2}\right)' = -\frac{2x}{(1+x^2)^2};$$

$$s''(x) \equiv \left(\sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1}\right)' = \sum_{n=1}^{\infty}(-1)^n 2n(2n-1)x^{2n-2} =$$

$$= 2\sum_{n=1}^{\infty}(-1)^n n(2n-1)x^{2n-2} = \left(-\frac{2x}{(1+x^2)^2}\right)' = \frac{6x^2-2}{(1+x^2)^3};$$

Bu yerda $\sum\limits_{n=1}^{\infty}(-1)^n n(2n-1)x^{2n-2}$ qatorning $x = 1/2$ nuqtadagi yig‘indisi izlanayotgan sonli qatorning yig‘indisiga teng:

$$\sum_{n=1}^{\infty}(-1)^n n(2n-1)\left(\frac{1}{2}\right)^{2n-2} = \sum_{n=1}^{\infty}(-1)^n n(2n-1)\frac{1}{2^{2n-2}} =$$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{n(2n-1)}{4^{n-1}}.$$

Ikkinchi tomondan

$$s''\left(\frac{1}{2}\right) = \left.\frac{6x^2-2}{(1+x^2)^3}\right|_{x=\frac{1}{2}} = -\frac{32}{125}.$$

U holda

$$\sum_{n=1}^{\infty}(-1)^n \frac{n(2n-1)}{4^{n-1}} = \frac{1}{2}s''\left(\frac{1}{2}\right) = -\frac{16}{125}$$

qiymatni hosil qilamiz. ◄

11.9. Teylor va Makloren qatorlari

Biz III bobda agar $f(x)$ funksiya $x = a$ nuqtada va uning biror atrofida $n+1$ tartibli va ungacha bo‘lgan barcha tartibli hosilalari mavjud bo‘lsa, n −tartibli

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}$$

Teylor formulasini hosil qilgan edik. Bu yerda

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

yig‘indi Teylor ko‘phadi va oxirgi

$$R_n(x) = \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}, 0 < \theta < 1$$

had Lagranj shaklidagi qoldiq hadi deb ataladi.

Agar $f(x)$ funksiyaning $x = a$ nuqtada va uning biror atrofida ixtiyoriy tartibli hosilasi mavjud bo‘lsa, bu atrofdan olingan ixtiyoriy x nuqtada $\lim\limits_{n\to\infty} R_n(x) = 0$ bo‘ladi. Shuning uchun yuqorida keltirilgan Teylor formulasidan $f(x)$ funksiyaning $(x-a)$ ifodaning darajalari bo‘yicha

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad (11.38)$$

yoyilmasini hosil qilamiz.

11.16-Ta’rif. $f(x)$ funksiya $x = a$ nuqtada va uning biror atrofida ixtiyoriy tartibli hosilalari mavjud bo‘lsin. U holda $f(x)$ funksiyani

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

ko‘rinishda tasvirlash uni Teylor qatoriga yoyish, $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ qatorning o‘zi esa $f(x)$ funksiyaning $x = a$ nuqtadagi Teylor qatori deb ataladi.

Agar $a = 0$ bo‘lsa, $f(x)$ funksiyaning Teylor qatoriga yoyilmasi Makloren qatori deb ataluvchi quyidagi

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{1!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \quad (11.39)$$

ko‘rinishni oladi.

11.29-Teorema. $f(x) = \sum\limits_{k=0}^{\infty} c_n(x-a)^n$ (C) darajali qator biror $(a - R; a + R)$ oraliqda $f(x)$ funksiyaga yaqinlashsin. U holda bu qator Teylor qatori, ya’ni

$$c_n = \frac{f^{(n)}(a)}{n!} \quad (11.40)$$

tengliklar o‘rinli bo‘ladi.

►(C) tenglikda $x = a$ qiymatni beramiz va natijada $c_0 = f(a)$ tenglik hosil bo‘ladi. So‘ngra (C) tenglikni hadma-had n marta differensiallaymiz:

$$f^{(n)}(x) = n!\,c_n + \frac{(n+1)!}{1!}c_{n+1}(x-a) + \frac{(n+2)!}{2!}c_{n+2}(x-a)^2 + \cdots$$

Bu yerda ham $x = a$ qiymatni berib $f^{(n)}(a) = n!\,c_n$ tenglikni va bundan (11.40) tenglikni hosil qilamiz.◄

$f(x)$ funksiyaning $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$ Teylor qatori qachon yaqinlashuvchi bo‘ladi va u yaqinlashuvchi bo‘lsa, $f(x)$ funksiyaga yaqinlashadimi degan tabiiy savol tug‘iladi. Bu savolga quyidagi teorema javob beradi.

11.30-Teorema. $f(x)$ funksiyaning $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$ Teylor qatori x nuqtada $f(x)$ qiymatga yaqinlashishi uchun qatorning $R_n(x)$ qoldiq hadi $n \to \infty$ bo‘lganda nolga intilishi, ya’ni $\lim\limits_{n\to\infty} R_n(x) = 0$ tenglikning o‘rinli bo‘lishi zarur va yetarli.

►**Zarurligi.** $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$ Teylor qatori x nuqtada $f(x)$ qiymatga yaqinlashsin, ya’ni $\lim\limits_{n\to\infty} s_n(x) = f(x)$. Bu yerda n −qismiy $s_n(x)$ yig‘indi

$$s_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k = P_n(x)$$

Teylor ko‘phadidan iborat. U holda

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty}[f(x) - P_n(x)] = \lim_{n\to\infty}[f(x) - s_n(x)] =$$
$$= f(x) - \lim_{n\to\infty} s_n(x) = f(x) - f(x) = 0.$$

Yetarliligi. $\lim\limits_{n\to\infty} R_n(x) = 0$ bo‘lsin. U holda

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) =$$
$$= f(x) - 0 = f(x).$$ ◄

Funksiyani darajali qatorga yoyish mumkinligining amaliyotda qo‘llash qulay bo‘lgan yetarli sharti quyidagi teoremada keltirilgan.

11.31-Teorema. $f(x)$ funksiya $(a-R; a+R)$ oraliqda ixtiyoriy tartibli hosilaga ega va shunday $M > 0$ soni topilib, barcha $x \in (a-R; a+$

R) nuqtalarda va barcha n natural sonlarda $\left|f^{(n)}(x)\right| \leq M$ tengsizliklar o‘rinli bo‘lsin. U holda $f(x)$ funksiya $(a-R; a+R)$ oraliqda Teylor qatoriga yoyiladi:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

►11.30-Teoremaga ko‘ra $\lim_{n\to\infty} R_n(x) = 0$ tenglikning bajarilishini ko‘rsatish yetarli. Teorema shartidan

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left|\frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}\right| \leq$$

$$\leq \lim_{n\to\infty} M \cdot \left|\frac{(x-a)^{n+1}}{(n+1)!}\right| = M \cdot \lim_{n\to\infty} \left|\frac{(x-a)^{n+1}}{(n+1)!}\right|$$

tengsizlini hosil qilamiz. Ko‘rinib turibdiki $\lim_{n\to\infty} R_n(x) = 0$ tenglikni ko‘rsatish uchun $\lim_{n\to\infty}\left|\frac{(x-a)^{n+1}}{(n+1)!}\right| = 0$ ekanligini ko‘rsatish yetarli. Buning uchun $\sum_{k=0}^{\infty}\left|\frac{(x-a)^{n+1}}{(n+1)!}\right|$ qatorni qaraymiz. Bu qator uchun

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{|x-a|^{n+2}}{(n+2)!}}{\frac{|x-a|^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{|x-a|^{n+2}(n+1)!}{|x-a|^{n+1}(n+2)!} =$$

$$= |x-a| \cdot \lim_{n\to\infty} \frac{1}{n+2} = 0 < 1$$

tengsizlik o‘rinli. U holda Dalamber alomatiga ko‘ra bu qator butun son o‘qida qator yaqinlashuvchi. Shuning uchun qator yaqinlashishining zaruriy shartiga ko‘ra

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

va demak $\lim_{n\to\infty} R_n(x) = 0$ bo‘ladi.◄

Ayrim elementar funksiyalarni Makloren qatoriga yoyish.

1. $f(x) = e^x$.

Bu funksiya ixtiyoriy $(-R; R)$ oraliqda barcha tartibli hosilalarga ega va bu hosilalar uchun

$$|f^{(n)}(x)| = e^x < e^R, n = 0,1,2, \dots$$

tengsizliklar o‘rinli. Shuning uchun 11.31-Teoremaga ko‘ra $f(x) = e^x$ funksiya ixtiyoriy $(-R; R)$ oraliqda va demak butun $\mathbb{R}$ son o‘qida Makloren qatoriga yoyiladi. Bundan tashqari $f^{(n)}(0) = e^0 = 1$. U holda bu funksiyaning Makloren qatori

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

ko‘rinishda bo‘ladi.

Bu qatorning yaqinlashish radiusi

$$R = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = +\infty.$$

2. $f(x) = e^{-x}$.

e^x funksiyaning yoyilmasida x o‘rniga $-x$ ifodani qo‘yib

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

Makloren qatorini hosil qilamiz.

3. $f(x) = \sin x$.

Bu funksiya butun $\mathbb{R}$ son o‘qida ixtiyoriy tartibli hosilaga ega va ular

$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), n = 1,2, \ldots$$

tengliklar bilan aniqlanadi. Hosilalarning $x = 0$ nuqtadagi qiymatlari esa

$$f^{(n)}(0) = \begin{cases} 0, & n = 2k; \\ (-1)^{k-1}, & n = 2k-1. \end{cases} \quad n = 1,2, \ldots$$

tenglikni qanoatlantiradi. Shunday qilib, $f(x) = \sin x$ funksiyaning Makloren qatori $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$ ko‘rinishda bo‘ladi. Bu funksiya butun $\mathbb{R}$ son o‘qida ixtiyoriy tartibli hosilaga ega va bu hosilalar $|f^{(n)}(x)| \leq 1$ tengsizliklarni qanoatlantiradi. Shuning uchun 11.31-Teoremaga ko‘ra $f(x) = \sin x$ funksiya $\mathbb{R}$ son o‘qida Makloren qatoriga yoyiladi:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, x \in \mathbb{R}.$$

Bu qatorning yaqinlashish radiusi $R = +\infty$ bo‘ladi.

4. $f(x) = \cos x$.

$f(x) = \sin x$ funksiyaning Makloren qatorini hadma-had differensiallab $f(x) = \cos x$ funksiyaning Makloren qatorini hosil qilamiz:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, x \in \mathbb{R}.$$

5. $f(x) = \operatorname{ch} x$, $f(x) = \operatorname{sh} x$ giperbolik kosinus va giperbolik sinus.

Bu funksiyalar $\operatorname{ch} x = (e^x + e^{-x})/2$; $\operatorname{sh} x = (e^x - e^{-x})/2$ tengliklar bilan aniqlanadi. Shuning uchun e^x va e^{-x} funksiyalarning yoyilmalarini hadma-had qo‘shib va ayirib

$$\operatorname{ch} x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, x \in \mathbb{R};$$

$$\operatorname{sh} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, x \in \mathbb{R}$$

Makloren qatorlarini hosil qilamiz.

6. $f(x) = (1+x)^\alpha$, bu yerda α −ixtiyoriy haqiqiy son

Bu funksiyani Makloren qatoriga yoyish uchun uning quyida keltiriladigan muhim bir xossasidan foydalanamiz:

$$f'(x) = ((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1} = \alpha \frac{(1+x)^\alpha}{1+x} = \alpha \frac{f(x)}{1+x}$$

yoki

$$(1+x)f'(x) = \alpha f(x). \qquad (11.41)$$

Endi bevosita berilgan funksiyaning Makloren qatorini tuzamiz. Faraz qilaylik bu funksiya $f(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} c_n x^n$ Makloren qatoriga yoyilgan bo‘lsin. c_n −hozircha noma’lum koeffisiyentlar va biz ularni topamiz. Buning uchun bu qatorni hadma-had differensiallaymiz:

$$f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}.$$

$f(x)$ va $f'(x)$ funksiyalarning Makloren qatorlarini (11.41) tenglikga qo‘yamiz:

$$(1+x)\sum_{n=1}^{\infty} c_n n x^{n-1} = \alpha \sum_{n=0}^{\infty} c_n x^n$$

yoki

$$\sum_{n=0}^{\infty}[nc_n+(n+1)c_{n+1}]x^n=\sum_{n=0}^{\infty}\alpha c_n x^n.$$

So‘ngi tenglikda x o‘zgaruvchining bir xil darajalari oldidagi koeffisiyentlarni tenglashtirib

$$c_1=\alpha c_0,$$
$$c_1+2c_2=\alpha c_1,$$
$$2c_2+3c_3=\alpha c_2,$$
$$\dots\dots\dots\dots\dots\dots,$$
$$(n-1)c_{n-1}+nc_n=\alpha c_{n-1},$$
$$\dots\dots\dots\dots\dots\dots$$

rekurient tenglamalar sistemasini hosil qilamiz va uning tenglamalarini ketma-ket yechamiz:

$$c_1=\alpha c_0,$$
$$c_2=\frac{(\alpha-1)c_1}{2}=\frac{(\alpha-1)\alpha c_0}{2!},$$
$$c_3=\frac{(\alpha-2)c_2}{3}=\frac{(\alpha-2)}{3}\frac{(\alpha-1)\alpha c_0}{2!}=\frac{(\alpha-2)(\alpha-1)\alpha c_0}{3!},$$
$$\dots\dots\dots\dots\dots\dots,$$
$$c_n=\frac{(\alpha-n+1)c_{n-1}}{n}=\frac{(\alpha-n+1)(\alpha-n+2)\dots\alpha c_0}{n!}.$$

$(1+x)^\alpha=\sum_{n=0}^{\infty}c_n x^n$ tenglikga $x=0$ qiymatni bersak $c_0=1$ qiymatni hosil qilamiz. Bu hosil qilingan qiymatni yuqoridagi tengliklarga qo‘yamiz va unda ketme-ket noma’lum c_n koffisiyentlarni topamiz:

$$c_1=\alpha,$$
$$c_2=\frac{\alpha(\alpha-1)}{2},$$
$$\dots\dots\dots\dots\dots\dots,$$
$$c_n=\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!},$$
$$\dots\dots\dots\dots\dots\dots$$

c_n koeffisiyentlarning bu topilgan qiymatlarini $(1+x)^\alpha$ funksiyaning Makloren yoyilmasiga qo‘yib

$$(1+x)^\alpha=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n$$

tenglikni hosil qilamiz. Hosil qilingan qator *binomial* qator va uning koeffisiyentlari *binomial koeffisiyentlar* deb ataladi. Bu qatorning yaqinlashish radiusini topamiz:

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}}{\dfrac{\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)}{(n+1)!}}\right|=$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)!\cdot\alpha(\alpha-1)\dots(\alpha-n+1)}{n!\cdot\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)}\right|=\lim_{n\to\infty}\left|\frac{(n+1)}{(\alpha-n)}\right|=1.$$

Shunday qilib binomial qator $(-1;1)$ oraliqda yaqinlashar ekan. Uning oraliq chegaralarida yaqinlashishini tekshirishni o‘quvchiga havola qilamiz.

Bu qator uchun muhim xususiy $\alpha=-1$ bo‘lganda binomial qator

$$\frac{1}{1+x}=1-x+x^2-x^3+\dots+(-1)^n x^n+\cdots,x\in(-1;1)$$

ko‘rinishda bo‘ladi. Bu tenglikda x o‘rniga $-x$ qo‘ysak

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots+x^n+\cdots,x\in(-1;1)$$

binomial qatorni hosil qilamiz.

7. $f(x)=\ln(1+x)\,,x>-1.$

Bu funksiyaning Makloren qatorini hosil qilish uchun $\int\limits_0^x\frac{1}{1+t}dt=\ln(1+x)$ tenglikdan foydalanamiz, ya’ni $\frac{1}{1+t}$ funksiyaning binomial yoyilmasini $[0;x],x\in(-1;1)$ kesmada hadma-had integrallaymiz:

$$\ln(1+x)=\int\limits_0^x(1-t+t^2-t^3+\dots+(-1)^n t^n+\cdots)dt$$

yoki

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+(-1)^n\frac{x^n}{n}+\cdots,x\in(-1;1).$$

Bu tenglikda x o‘rniga $-x$ qo‘ysak

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots-\frac{x^n}{n}-\cdots,x\in(-1;1).$$

Makloren qatorini hosil qilamiz.

$\ln(1+x)$ funksiyaning Makloren qatoridan $\ln(1-x)$ funksiyaning qatorini hadma-had ayirsak

$$\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right), x\in(-1;1)$$

yana bir Makloren qatorini hosil qilamiz.

11.10. Furye qatorlari

Radiotexnika, elektronika, avtomatik boshqarish nazariyasi va boshqa shunga o‘xshash faoliyat turlarida uchraydigan, ma’lum vaqt oraliqlarida takrorlanuvchi davriy jarayonlarni ifodalovchi davriy funksiyalarni darajali qatorlarga emas, balki trigonometrik qatorlar deb ataluvchi qatorlarga yoyish maqsadga muvofiq bo‘ladi.

Davriy funksiyalarga II bobda ta’rif bergan edik. Shu ta’rifni eslatib o‘tamiz.

11.17-Ta’rif. $X\subseteq\mathbb{R}$ to‘plamda aniqlangan $f(x)$ funksiyaning *davri* deb, shunday $T>0$ songa aytiladiki, bunda ixtiyoriy $x\in X$ nuqta uchun $(x-T, x+T)\in X$ munosabat va

$$f(x+T)=f(x-T)=f(x) \qquad (11.42)$$

tenglik oriinli bo‘ladi.

Davrga ega bo‘lgan funksiyani *davriy funksiya*, (11.42) tebglikni qanoatlantiruvchi T sonlarning eng kichigiga esa $f(x)$ funksiyaning *davri* deb ataymiz.

11.18-Ta’rif. Trigonometrik qator deb

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\cdots+a_n\cos nx+$$

$$+b_n\sin nx+\cdots=a_0/2+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (11.43)$$

ko‘rinishdagi funksional qatorga aytiladi, bu yerda $a_0, a_1, b_1, \ldots$ o‘zarmas sonlar (11.43) *trigonometrik qatorning koeffisiyentlari* deb ataladi.

a_n va b_n koffisiyentlarni hisoblashdagi bir xillikni saqlash maqsadida (11.43) formulada ozod ahad $a_0/2$ ko‘rinishda yozilgan.

(11.43) trigonometrik qatorning $s_n(x)$ qismiy yig‘indilari *trigonometrik sistema* deb ataluvchi

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

funksiyalar sistemasidan olingan funksiyalarning chiziqli kombinatsiyalaridan iborat. (11.43) qatorning hadlari davri 2π bo‘lgan davriy funksiyalar bo‘lganligi uchun, agar qator yaqinlashuvchi bo‘lsa, uning $s(x)$ yig‘indisi ham davri $T=2\pi$ bo‘lgan davriy funksiya bo‘ladi:

$$s(x+2\pi)=s(x), x\in(-\infty;\ +\infty).$$

11.18-Ta'rif. Davri 2π bo'lgan $f(x)$ funksiyani (11.43) trigonometrik qatorga yoyish deganda yig'indisi $f(x)$ funksiyaga teng bo'lgan trigonometrik qatorni topish tushuniladi.

Bizga kerak bo'ladigan ba'zi formulalarni keltiramiz:

$$\int_{-\pi}^{\pi}\cos nx\,dx=\frac{\sin nx}{n}\bigg|_{-\pi}^{\pi}=0\ \ (n\neq 0) \tag{11.44}$$

$$\text{ixtiyoriy } n \text{ uchun } \int_{-\pi}^{\pi}\sin nx\,dx=-\frac{\cos nx}{n}\bigg|_{-\pi}^{\pi}=0 \tag{11.45}$$

$$\int_{-\pi}^{\pi}\cos mx\cdot\cos nx\,dx=\frac{1}{2}\left(\int_{-\pi}^{\pi}\cos(m+n)x+\cos(m-n)x\right)dx=$$
$$=\begin{cases}0 & (m\neq n),\\ \pi & (m=n),\end{cases} \tag{11.46}$$

$$\int_{-\pi}^{\pi}\sin mx\cdot\cos nx\,dx=$$
$$\frac{1}{2}\left(\int_{-\pi}^{\pi}\sin(m+n)x+\sin(m-n)x\right)dx=0, \tag{11.47}$$

$$\int_{-\pi}^{\pi}\sin mx\cdot\sin nx\,dx=\frac{1}{2}\left(\int_{-\pi}^{\pi}\cos(m-n)x-\cos(m+n)x\right)dx=$$
$$=\begin{cases}0 & (m\neq n),\\ \pi & (m=n),\end{cases} \tag{11.48}$$

Shunday qilib, (11.44)-(11.48) tengliklarga ko'ra yuqorida kiritilgan

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

trigonometrik sistemaga kirgan ixtiyoriy ikkita funksiya ko'paytmasining uzunligi 2π bo'lgan kesma bo'yicha olingan aniq integrali nolga teng, ya'ni bu sistema *ortoganallik xossasiga* ega ekan.

Endi asosiy masalani qo'yamiz.

Davri 2π bo'lgan $f(x)$ funksiya berilgan bo'lsin. Qanday shartlar bajarilganda $f(x)$ funksiyaga yaqinlashuvchi (11.43) ko'rinishdagi trigonometrik qatorni topish mumkin?

11.32-Teorema. $[-\pi;\pi]$ kesmadan olingan ixriyoriy x uchun

$$f(x)=a_0/2+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{11.49}$$

tenglik o'rinli bo'lsin va bunda tenglikning o'ng tomonidagi qator bu kesmada tekis yaqinlashsin. U holda

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \ \ (n = 0,1,2,\dots),$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \ \ (n = 1,2,\dots) \qquad (11.50)$$
tengliklar o‘rinli bo‘ladi.

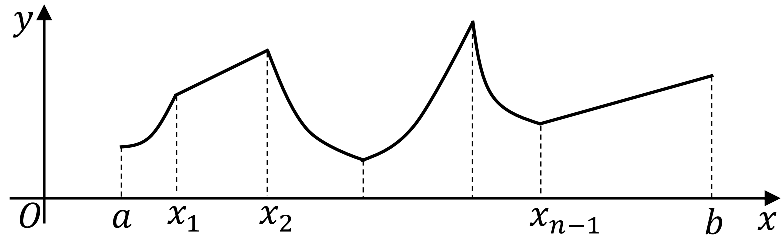
►Teorema shartiga ko‘ra (11.49) tenglikning o‘ng tomonidagi qator $[-\pi;\pi]$ kesmada tekis yaqinlashadi. Bu esa 11.21-Teoremaga ko‘ra $f(x)$ funksiyaning uzluksizligini va demak integrallanuvchiligini ta’minlaydi. Bundan tashqari 11.22-Teoremaga ko‘ra (11.49) qatorni hadma-had integrallash mumkin:
$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right)$$
yoki
$$\int_{-\pi}^{\pi} f(x)dx = a_0\pi,$$
ya’ni (11.50) formulalarning birinchisi $n = 0$ hol uchun kelib chiqadi.

(11.49) tenglikni navbatma-navbat $\cos mx$ va $\sin mx$ funksiyalarga ko‘paytiramiz. Hosil bo‘lgan qatorlar ham $[-\pi;\pi]$ kesmada tekis yaqinlashadi, shuning uchun ularni ham hadma-had integrallash mumkin. (11.44)-(11.48) formulalarni inobatga olsak bu integrallashlardan so‘ng
$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\pi,\ n = 1,2,3,\dots$$
$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n\pi,\ n = 1,2,3,\dots$$
tengliklarga ega bo‘lamiz. Ular esa o‘z navbatida (11.50) tengliklarni ta’minlaydi.◄

11.19-Ta’rif. (11.50) tengliklar bilan aniqlanuvchi $a_0, a_1, a_2, \dots$, $b_1, b_2, \dots$ koeffisiyentlar $f(x)$ funksiyaning Furye koeffisiyentlari va (11.49) trigonometrik qator $f(x)$ funksiyaning Furye qatori deb ataladi.

Shunday qilib, $[-\pi;\pi]$ kesmada integrallanuvchi ixtiyoriy $f(x)$ funksiyaga koeffisiyentlari (11.50) tengliklar bilan aniqlanuvchi Furye qatori mos qo‘yildi:
$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

Qanday shartlar bajarilganda bu moslik belgisini tenglik bilan almashtirish mumkin degan savolga javob beramiz.



11.2-rasm

11.20-Ta’rif. Agar chekli sondagi shunday $a < x_1 < x_2 < \cdots < x_{n-1} < b$ nuqtalar topilib, bu nuqtalardan tuzilgan $(a; x_1), (x_1; x_2),\ \ldots, (x_{n-1}; b)$ oraliqlarning har birida $f(x)$ funksiya monoton bo‘lsa, $f(x)$ funksiya $[a; b]$ kesmada bo‘lakli monoton deyiladi (11.2-rasm).

11.38-Misol. $f(x) = x^3 - 3x^2 - 9x + 1$ funksiya $(-\infty; +\infty)$ oraliqda bo‘lakli monoton bo‘ladi. Bu oraliqni $(-\infty; -1)$, $(-1; 3)$, $(3; +\infty)$ oraliqlarga ajratamiz. Berilgan funksiya bu oraliqlarning birinchisida o‘sadi, ikkinchisida kamayadi va uchinchisida yana o‘sadi.

11.39-Misol. $f(x) = \sin x$ funksiya $(0; 2\pi)$ oraliqda bo‘lakli monoton. Bu oraliqni $(0; \pi/2)$, $(\pi/2; 3\pi/2)$ va $(3\pi/2; 2\pi)$ oraliqlarga ajratsak, $f(x) = \sin x$ funksiya bu oraliqlarning har birida monoton, ya’ni ulardan birinchisida o‘sadi, ikkinchisida kamayadi va uchinchisida yana o‘sadi.

Agar $f(x)$ funksiya $[a; b]$ kesmada bo‘lakli monoton bo‘lsa, u holda kesmaning ixtiyoriy ichki c nuqtasida chekli chap va o‘ng limitlar mavjud:

$$\lim_{x \to c-0} f(x) = f(c - 0),$$

$$\lim_{x \to c+0} f(x) = f(c + 0).$$

Biz funksiyaning Furye qatoriga yoyilishi uchun yetarli shartni ifodalovchi Dirixle teoremasini isbotsiz keltiramiz.

11.33-Teorema. Agar davri 2π bo‘lgan davriy $f(x)$ funksiya $[-\pi; \pi]$ kesmada bo‘lakli uzluksiz, ya’ni uzluksiz yoki chekli sondagi I tur uzilish nuqtalariga ega va bo‘lakli monoton bo‘lsa, uning Furye qatori kesmaning ixtiyoriy x nuqtasida yaqinlashadi.

Agar $S(x) -$ bu qatorning yig‘indisi bo‘lsa, $f(x)$ funksiyaning uzluksizlik nuqtalarida

$$S(x) = f(x),$$

uzilish nuqtalarida esa

$$S(x) = \frac{1}{2}\left(f(x-0) + f(x+0)\right) \qquad (11.51)$$

va bundan tashqari

$$S(\pi) = S(-\pi) = \frac{1}{2}\left(f(-\pi+0) + f(\pi-0)\right) \qquad (11.52)$$

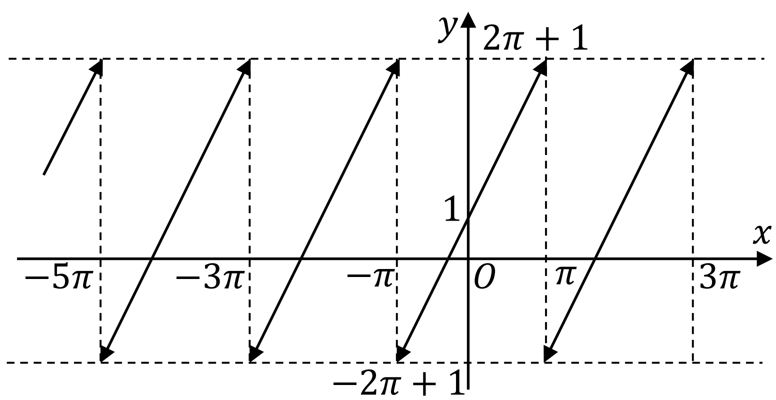
tengliklar o‘rinli bo‘ladi.

Teoremada keltirilgan funksiyaning bo‘lakli uzluksizlik va bo‘lakli monotonlik shartlari Dirixle shartlari deb ataladi. Shunday qilib $[-\pi;\pi]$ kesmada Dirixle shartlarini qanoatlantiruvchi $f(x)$ funksiya uchun uzluksizlik nuqtalarida

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

tenglik o‘rinli bo‘lar ekan.

11.40-Misol. Davri 2π bo‘lgan va $(-\pi;\pi)$ kesmada $f(x) = 2x+1$ tenglik bilan aniqlanuvchi $f(x)$ funksiyani Furye qatoriga yoying.



11.3-rasm

►Bu funksiya Dirixle shrtlarini qanoatlantiradi (11.3-rasm). Shuning uchun uni Furye qatoriga yoyish mumkin. Bu qatorning Furye koeffisiyentlarini topamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)dx = \frac{x^2+x}{\pi}\bigg|_{-\pi}^{\pi} = 2,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)d\left(\frac{\sin nx}{n}\right) =$$

$$= \frac{(2x+1)\sin nx}{\pi n}\bigg|_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}\sin nx\, dx = \frac{2\cos nx}{\pi n^2}\bigg|_{-\pi}^{\pi} =$$

$$= \frac{2\cos n\pi - 2\cos(-n\pi)}{\pi n^2} = 0, n = 1,2, \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)d\left(-\frac{\cos nx}{n}\right) =$$
$$= -\frac{(2x+1)\cos nx}{\pi n}\Bigg|_{-\pi}^{\pi} + \frac{2}{\pi n}\int_{-\pi}^{\pi}\cos nx\,dx =$$
$$= -\frac{(2\pi+1)\cos n\pi}{\pi n} + \frac{(-2\pi+1)\cos(-n\pi)}{\pi n} + \frac{2\sin nx}{\pi n^2}\Bigg|_{-\pi}^{\pi} =$$
$$= -\frac{4}{n}\cos n\pi = -4\frac{(-1)^n}{n}.$$

Shunday qilib berilgan funksiya uchun Furye qatori

$$2x+1 = 1 - 4\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}, -\pi < x < \pi$$

ko‘rinishda bo‘lar ekan.◄

Juft va toq funksiyalarning Furye qatorlari. $f(x)$ funksiya $[-a;a]$ kesmada aniqlangan bo‘lsin . Agar ixtiyoriy $x \in [-a;a]$ nuqta uchun $f(-x) = f(x)$ tenglik o‘rinli bo‘lsa, funksiya bu kesmada juft, $f(-x) = -f(x)$ o‘rinli bo‘lsa, toq deyiladi.

Integrallanuvchi juft funksiyalar uchun

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \qquad (11.53)$$

toq funksiyalar uchun esa

$$\int_{-a}^{a} f(x)dx = 0 \qquad (11.54)$$

tengliklar o‘rinli bo‘ladi.

Ikkita juft va ikkita toq funksiyalarning ko‘paytmasi juft bo‘lishi, juft va toq funksiyaning ko‘paytmasi esa toq bo‘lishi ravshan. Ma’lumki $\cos nx$ juft, $\sin nx$ esa toq funksiya. (11.53) va (11.54) tengliklarni inobatga olsak,
juft funksiyalar uchun

$$\begin{gathered} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, (n = 0,1,\dots) \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0, (n = 1,2,\dots) \end{gathered} \qquad (11.55)$$

tengliklarga ega bo‘lamiz, ya’ni juft funksiyalarning Furye qatori faqat kosinuslardan iborat bo‘ladi:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx. \tag{11.56}$$

Toq funksiyalar uchun ham (11.53) va (11.54) tengliklarni inobatga olsak a_n va b_n koeffisiyentlar uchun

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0, (n = 0,1, \dots)$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx, (n = 1,2, \dots) \tag{11.56}$$

tengliklar o‘rinli bo‘ladi, ya’ni toq funksiyalarning Furye qatori faqat sinuslardan iborat bo‘lar ekan:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \tag{11.57}$$

11.41-Misol. $f(x) = x^2$ juft funksiyaning $[-\pi;\pi]$ kesmada kosinuslardan iborat Furye qatorini tuzing.

►(11.55) formulalarga ko‘ra a_n koeffisiyentlarni topamiz:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\bigg|_0^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 d\left(\frac{\sin nx}{n}\right) = \frac{2}{\pi}\frac{x^2\sin nx}{3n}\bigg|_0^{\pi} -$$
$$-\frac{2}{\pi n}\int_0^{\pi} 2x\sin nx\, dx = -\frac{4}{\pi n}\int_0^{\pi} xd\left(-\frac{\cos nx}{n}\right)dx = \frac{4}{\pi n}\frac{x\cos nx}{n}\bigg|_0^{\pi} -$$
$$-\frac{4}{\pi n^2}\int_0^{\pi} \cos nx\, dx = \frac{4}{\pi n}\frac{\pi\cos n\pi}{n} - \frac{4}{\pi n^2}\frac{\sin nx}{n}\bigg|_0^{\pi} = (-1)^n\frac{4}{n^2}.$$

Shunday qilib, qidirilayotgan Furye qatori

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} \tag{11.58}$$

ko‘rinishda bo‘lar ekan.◄

11.42-Misol. Qator yig‘indisini hisoblang

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

►Bu qator yig‘indisini S orqali belgilaymiz. (11.58) tenglikda $x = 0$ deb olamiz. U holda

$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos 0}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

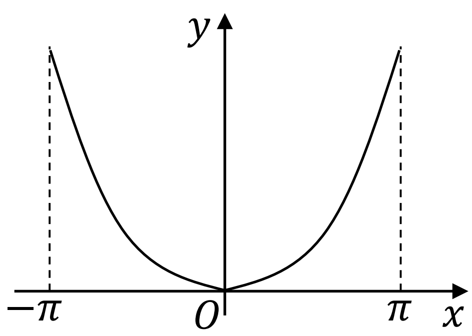
So‘ngi tenglikning o‘ng tomonidagi qator izlanayotgan qatorning yig‘indisini beradi, ya’ni

$$\frac{\pi^2}{3}-4S=0.$$

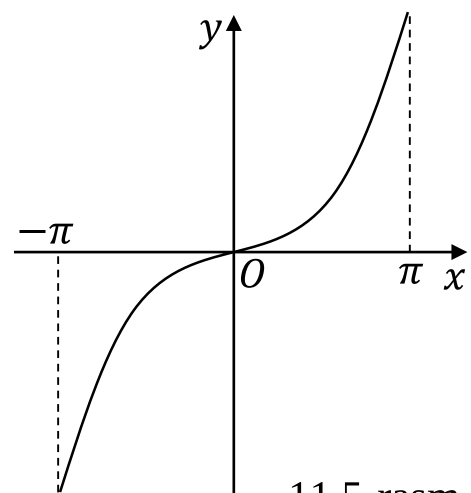
Bundan esa qidirilayotgan sonli qator yig‘indisini topamiz:

$$\sum_{n=1}^{\infty}(-1)^{n+1}/n^2=\pi^2/12.◄$$

Funksiyani $[0;\pi]$ kesmada sinuslar va kosinuslar bo‘yicha yoyish. $f(x)$ funksiya faqat $[0;\pi]$ kesmada berilgan bo‘lsin. $f(x)$ funksiyani bu kesmada Furye qatoriga yoyish uchun uni $[-\pi;0]$ kesmada davom ettiramiz. Natijada $[-\pi;\pi]$ kesmada aniqlangan funksiyaga ega bo‘lamiz va uni bu kesmada Furye qatoriga yoyishimiz mumkin.



11.4-rasm



11.5-rasm

Bizni $f(x)$ funksiyaning $[0;\pi]$ kesmadagi qiymatlari qiziqtirgani uchun funksiyani $[-\pi;0]$ kesmada bizga “qulay” bo‘ladigan qilib davom ettirishimiz mumkin. Natijada hosil bo‘lgan funksiyaning Furye qatori ham turlicha bo‘ladi. $f(x)$ funksiyani $[-\pi;0]$ kesmada turli variantlarda aniqlashimiz mumkin. Biz ulardan ikkitasida to‘xtalamiz.

1. $f(x)$ funksiyani juft bo‘ladigan qilib davom ettiramiz, ya’ni

$$f(-x)=f(x),0\le x\le\pi$$

tenglik o‘rinli bo‘lsin (11.4-rasm). Bu holda yangi juft funksiyaning Furye qatori faqat kosinuslardan iborat bo‘ladi.

2. $f(x)$ funksiyani toq bo‘ladigan qilib davom ettiramiz, ya’ni

$$f(-x)=-f(x),0\le x\le\pi$$

tenglik o‘rinli bo‘lsin (11.5-rasm). Bu holda yangi toq funksiyaning Furye qatori faqat sinuslardan iborat bo‘ladi.

11.43-Misol. $f(x) = x, 0 \le x \le \pi$ funksiyani i) kosinuslar, ii) sinuslar bo‘yicha Furye qatoriga yoying.

►i) Berilgan funksiyani kosinuslar boyicha yoyish uchun

$$f(x) = \begin{cases} x, 0 \le x \le \pi, \\ -x, -\pi \le x \le 0 \end{cases}$$

tenglik bilan juft funksiyani tuzamiz va uning Furye koeffisiyentlarini hisoblaymiz:

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \left.\frac{x^2}{\pi}\right|_0^\pi = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\, dx = \frac{2}{\pi}\int_0^\pi x d\left(\frac{\sin nx}{n}\right) = \frac{2}{\pi}\left.\frac{x\sin nx}{n}\right|_0^\pi -$$

$$-\frac{2}{\pi n}\int_0^\pi \sin nx\, dx = \frac{2}{\pi n}\left.\frac{\cos nx}{n}\right|_0^\pi = \frac{2}{\pi n^2}(\cos n\pi - \cos 0) =$$

$$= \frac{2}{\pi n^2}((-1)^n - 1) = \begin{cases} -\dfrac{4}{\pi n^2}, n = 2m-1, \\ 0, \qquad n = 2m, \end{cases}$$

ya’ni kosinuslar qatorida toq hadlar qatnashadi:

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

ii) Berilgan funksiyani sinuslar bo‘yicha yoyish uchun $f(x) = x, -\pi \le x \le \pi$ tenglik bilan toq funksiyani tuzamiz va uning Furye koeffisiyentlarini hisoblaymiz:

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\, dx = \frac{2}{\pi}\int_0^\pi x d\left(-\frac{\cos nx}{n}\right) = -\frac{2}{\pi}\left.\frac{x\cos nx}{n}\right|_0^\pi +$$

$$+\frac{2}{\pi n}\int_0^\pi \cos nx\, dx = -\frac{2}{\pi}\frac{\pi\cos n\pi}{n} + \frac{2}{\pi n}\left.\frac{\sin nx}{n}\right|_0^\pi = \frac{2}{n}(-1)^{n+1},$$

ya’ni sinuslar qatori

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

ko‘rinishda bo‘ladi.◄

Ixtiyoriy davrli funksiyani Furye qatoriga yoyish. Davri 2π bo‘lmagan davriy funksiyalarni ham Furye qatoriga yoyish mumkin.

$f(x)$ davri $2l, l > 0$ bo‘lgan davriy funksiya bo‘lsin va u $[-l; l]$ kesmada Dirixle shartlarini qanoatlantirsin. Bu funksiyani $[-l; l]$ kesmada Furye qatoriga yoyish uchun $x = \frac{l}{\pi} t$ deb yangi t o‘zgaruvchini kiritamiz. U holda $\varphi(t) = f\left(\frac{l}{\pi} t\right)$ funksiya davri 2π bo‘lgan davriy funksiya bo‘ladi:

$$\varphi(t + 2\pi) = f\left[\frac{l}{\pi}(t + 2\pi)\right] = f\left(\frac{l}{\pi} t + 2l\right) = f\left(\frac{l}{\pi} t\right) = \varphi(t).$$

Shuning uchun uni $[-\pi; \pi]$ kesmada Furye qatoriga yoyish mumkin bo‘ladi:

$$f\left(\frac{l}{\pi} t\right) = \varphi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

va bu yerda

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(t) \cos nt \, dt \ \ (n = 0,1,2, \dots),$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(t) \sin nt \, dt \ \ (n = 1,2, \dots)$$

yoki

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{l}{\pi} t\right) \cos nt \, dt \ \ (n = 0,1,2, \dots),$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{l}{\pi} t\right) \sin nt \, dt \ \ (n = 1,2, \dots).$$

$t = \frac{\pi}{l} x, dt = \frac{\pi}{l} dx$ deb olib dastlabki x o‘zgaruvchiga qaytamiz:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right), \tag{11.59}$$

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \ \ (n = 0,1,2, \dots), \\ b_n &= \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \ \ (n = 1,2, \dots). \end{aligned} \tag{11.60}$$

Koeffisiyentlari (11.60) formulalarga ko‘ra hisoblanadigan (11.59) qator davri $2l$ bo‘lgan davriy $f(x)$ funksiya uchun Furye qatori deb ataladi.

11.44-Misol. $f(x) = 2x + 1$ funksiyani $[-3; 3]$ kesmada Furye qatoriga yoying.

►Berilgan funksiya $[-3; 3]$ kesmada Dirixle shartlarini to‘liq qanoatlantiradi. $l = 3$ hol uchun berilgan funksiyaning Furye koeffisiyentlarini (11.60) formulalarga ko‘ra hisoblaymiz:

$$a_0 = \frac{1}{3}\int_{-3}^{3}(2x+1)dx = \frac{1}{3}(x^2+x)\Big|_{-3}^{3} = \frac{1}{3}(9+3-9+3) = 2,$$

$$a_n = \frac{1}{3}\int_{-3}^{3}(2x+1)\cos\frac{n\pi x}{3}dx = \frac{1}{3}\int_{-3}^{3}(2x+1)d\left(\frac{3}{n\pi}\sin\frac{n\pi x}{3}\right) =$$

$$= \frac{1}{n\pi}(2x+1)\sin\frac{n\pi x}{3}\Big|_{-3}^{3} - \frac{2}{n\pi}\int_{-3}^{3}\sin\frac{n\pi x}{3}dx = \frac{2}{n\pi}\frac{3}{n\pi}\cos\frac{n\pi x}{3}\Big|_{-3}^{3} =$$

$$= \frac{6}{n^2\pi^2}(\cos n\pi - \cos(-n\pi)) = 0,$$

$$b_n = \frac{1}{3}\int_{-3}^{3}(2x+1)\sin\frac{n\pi x}{3}dx = \frac{1}{3}\int_{-3}^{3}(2x+1)d\left(-\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right) =$$

$$= -\frac{1}{n\pi}(2x+1)\cos\frac{n\pi x}{3}\Big|_{-3}^{3} - \frac{2}{n\pi}\int_{-3}^{3}\cos\frac{n\pi x}{3}dx =$$

$$= -\frac{1}{n\pi}(7\cos n\pi + 5\cos(-n\pi)) - \frac{2}{n\pi}\frac{3}{n\pi}\sin\frac{n\pi x}{3}\Big|_{-3}^{3} =$$

$$= -\frac{12}{n\pi}\cos n\pi = (-1)^{n+1}\frac{12}{n\pi}.$$

Shunday qilib $f(x) = 2x+1$ funksiyaning $[-3;3]$ kesmadagi Furye qatori

$$2x+1 = 1 + \frac{12}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\frac{n\pi x}{3}$$

ko‘runishda bo‘lar ekan.◄

Davriy bo‘lmagan funksiyani Furye qatoriga yoyish. Davriy bo‘lmagan $f(x)$ funksiya $\mathbb{R}$ butun sanoq o‘qida aniqlangan bo‘lsa, uni Furye qatoriga yoyib bo‘lmaydi, chunki Furye qatorining yig‘indisi davriy funksiya bo‘ladi.

Shuning uchun Davriy bo‘lmagan funksiyani Furye qatoriga yoyish masalasini ixtiyoriy chekli $[a;b]$ oraliqda qaraymiz.

$f(x)$ funksiya $[a;b]$ kesmada aniqlangan va u Dirixle shartlarini qanoatlantirsin. U holda

$$t = -\pi + \frac{x-a}{b-a}2\pi, \qquad x = a + \frac{b-a}{2\pi}(t+\pi) \qquad (11.61)$$

tengliklar bilan yangi o‘zgaruvchi kiritamiz. x o‘zgaruvchi $[a;b]$ kesmada o‘zgarganda t o‘zgaruvchi $[-\pi;\pi]$ kesmada o‘zgaradi.

U holda

$$\varphi(t) = f\left(a + \frac{b-a}{2\pi}(t+\pi)\right)$$

t o‘zgaruvchining funksiyasi sifatida Dirixle shartlarini qanoatlantiradi va uni $[-\pi;\pi]$ kesmada Furye qatoriga yoyish mumkin bo‘ladi. Hosil qilingan bu Furye qatorida t o‘rniga (11.61) tengliklarning birinchisini qo‘yib, dastlabki x o‘zgaruvchiga qaytamiz va natijada $f(x)$ funksiyaning $[a;b]$ kesmadagi Furye qatorini hosil qilamiz.

11.45-Misol. $f(x) = 4x + 1$ funksiyani $(2;4)$ oraliqda Furye qatoriga yoying.

► $t = (x-3)\pi,\ x = 3 + \frac{t}{\pi}$ deb yangi t o‘zgaruvchini kiritamiz va bunda x o‘zgaruvchi $(2;4)$ oraliqda o‘zgarganda t oz’garuvchi $(-\pi;\pi)$ oraliqda o‘zgaradi. U holda

$$\varphi(t) = f\left(3 + \frac{t}{\pi}\right) = 13 + \frac{4t}{\pi}$$

Funksiyaga ega bo‘lamiz va uni $(-\pi;\pi)$ oraliqda Furye qatoriga yoyamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(13 + \frac{4t}{\pi}\right)dt = 26;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(13 + \frac{4t}{\pi}\right)\cos nt\, dt = 0;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(13 + \frac{4t}{\pi}\right)\sin nt\, dt = (-1)^{n+1}\frac{8}{n\pi};$$

Shunday qilib,

$$13 + \frac{4t}{\pi} = 13 + \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin nt.$$

Bu tenglikda t o‘zgaruvchi o‘rniga $t = (x-3)\pi$ ifodani qo‘yamiz

$$4x + 1 = 13 + \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin n\pi(x-3).$$

O‘ng tomondagi $\sin n\pi(x-3)$ ifodani o‘zartiramiz:

$$\sin n\pi(x-3) = \sin n\pi x\cos(-3n\pi) - \sin(-3n\pi)\cos n\pi x =$$
$$= (-1)^n \sin n\pi x.$$

Hosil bo‘lgan ifodani yuqoridagi qatorga qo‘yamiz va qidirilayotgan Furye qatorini hosil qilamiz:

$$4x + 1 = 13 + \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(-1)^n\sin n\pi x$$

yoki

$$4x+1=13-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$$

Bu tenglik faqat $(2;4)$ oraliqdagina o‘rinli ekanligini ta’kidlab o‘tamiz.◄

Nazorat savollari

1. Sonli qatorga ta’rif bering.
2. Sonli qator qachon yaqinlashuvchi deyiladi?
3. Berilgan qatorning qoldiq qatori qanday aniqlanadi?
4. Qator yaqinlashishing zaruriy shartini bayon qiling.
5. Musbat hadli qatorlar yaqinlashishining qanday alomatlarini bilasiz?
6. Musbat hadli va ishoralari navbatlashuvchi qatorlarning farqini ayting.
7. Sonli qator qachon absolyut yaqinlashuvchi deyiladi?
8. Sonli qatorlarning absolyut va shartli yaqinlashishlari farqini ayting.
9. Funksional qatorlarga ta’rif bering.
10. Funksional qator qachon tekis yaqinlashuvchi deyiladi?
11. Darajali qatorlarni qanday sohada hadma-had integrallash mumkin?
12. Qanday qatorlarga Furye qatori deb ataladi?
13. Juft funksiyalarning Furye qatori qanday ko‘rinishda bo‘ladi?
14. Toq funksiyalarning Furye qatori qanday ko‘rinishda bo‘ladi?
15. Davriy bo‘lmagan funksiyalarni ham Furye qatoriga yoyish mumkinmi?

Mashqlar.

Sonli qatorning u_n umumiy hadi berilgan. Uning dastlabki to‘rtta hadini yozing.

1. $u_n=n/(2n+1)^2$; **2.** $u_n=(-1)^n\sin 2n$; **3.** $u_n=n/(5^n-2^n)$;

$u_1+u_2+u_3+u_4+\cdots$ sonli qatorning umumiy hadini toping.

4. $\frac{1}{2}+\frac{2}{5}+\frac{3}{9}+\frac{4}{17}+\cdots$ **5.** $\frac{2}{3}+\frac{8}{5}+\frac{26}{7}+\frac{80}{9}+\cdots$

Qator yig‘indisini toping.

6. $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$; **7.** $\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$; **8.** $\sum\limits_{n=1}^{\infty}\left(\frac{1}{3^n}+\frac{1}{2^{n+1}}\right)$;

Qator yaqinlashishini tekshiring.

9. $\sum\limits_{n=1}^{\infty}\frac{n^3+2n^2-3n+2}{n(n+1)}$; **10.** $\sum\limits_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$; **11.** $\sum\limits_{n=1}^{\infty}\frac{2^n\cdot n!}{n^n}$;

Qatorni absolyut va shartli yaqinlashishga tekshiring.

12. $\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{3^n+1}$; **13.** $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+1}}{n\sqrt{n}+1}$; **14.** $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$;

Darajali qatorning yaqinlashish sohasini toping.

15. $\sum_{n=1}^{\infty}\frac{x^n}{n!}$; **16.** $\sum_{n=1}^{\infty} n!\,x^n$; **17.** $\sum_{n=1}^{\infty} 5^n x^n$; **18.** $\sum_{n=1}^{\infty}\frac{(n+1)^5}{2n+1}(x+1)^n$;

19. $f(x)=e^x$ funksiyani $(-\pi;\pi)$ oraliqda Furye qatoriga yoying.

20. $f(x)=3x^2+2$ funksiyani $(-1;1)$ oraliqda Furye qatoriga yoying.

Javoblar

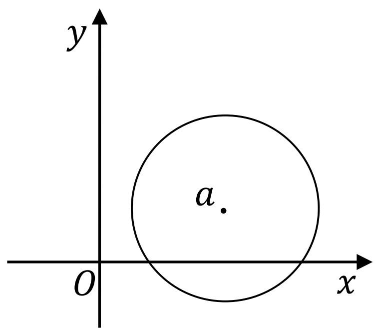
6. 1; **7.** $1/2$; **8.** 1; **9.** Uzoqlashadi; **10.** Yaqinlashadi; **11.** Yaqinlashadi; **12.** Absolyut yaqinlashadi; **13.** Shartli yaqinlashadi; **14.** Uzoqlashadi; **15.** $(-\infty;+\infty)$ oraliqda absolyut yaqinlashadi; **16.** $\{0\}$; **17.** $\left(-\frac{1}{5};\frac{1}{5}\right)$ oraliqda absolyut yaqinlashadi; **18.** $(-2;0)$ oraliqda absolyut yaqinlashadi; **19.** $e^x =$
$= \frac{1}{\pi}\operatorname{sh}\pi + \frac{2}{\pi}\operatorname{sh}\pi\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos nx}{n^2+1}+\frac{\sin nx}{n^2-1}\right)$; **20.** $3x^2+2=3+$
$+\frac{6}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$.

O‘ZGARUVCHILI FUNKSIYALAR NAZARIYASI ELEMENTLARI

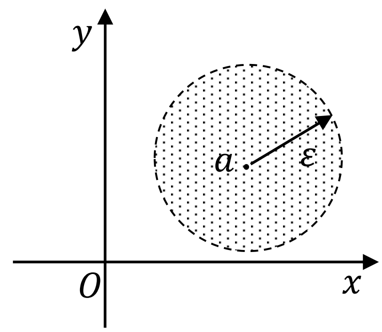
12.1. Kompleks tekislik nuqtalari to‘plami.

Kompleks sonlar va ular ustida asosiy amallarni bajarish haqida darslikning 1- qismida (1.4-mavzu) to‘xtab o‘tilgan edi. Shuningdek, bu mavzuda kompleks sonlar ketma- ketligi va uning limiti haqida tushuncha berilgan. Kompleks sonning modulini bilgan holda kompleks tekislikdagi ba’zi chiziq va sohalalarni ifodalash mumkin bo‘ladi.

1. $E = \{z: |z - a| = R\}$ to‘plam, kompleks tekislikda a nuqtadan bir xil R masofada joylashgan nuqtalar to‘plamidan iborat, bu yerda $R > 0$ va a berilgan nuqta. Bu to‘plam tekislikda markazi a nuqtada, radiusi R ga teng bo‘lgan aylanani ifodalaydi (12.1- rasm).



12.1-rasm



12.2-rasm

2. $E = \{z: |z - a| < \varepsilon\}$ to‘plam, markazi a nuqtada, radiusi $\varepsilon > 0$ bo‘lgan ochiq doiradan iborat. Bu to‘plamni ko‘pincha a nuqtaning ε atrofi deb ataladi (12.2- rasm).

3. $E = \{z: |z| > R\}$ to‘plam, markazi nolda radiusi R ga teng bo‘lgan doiraning tashqarisidan iborat. Bu to‘plam *cheksizlikning atrofi* deb ataladi.

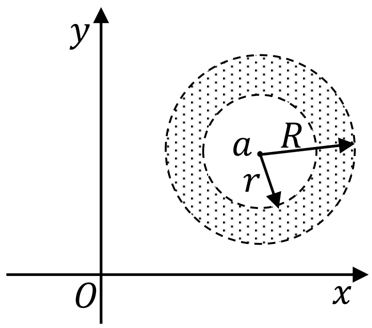
12.1-Ta’rif. Cheksiz uzoqlashgan nuqta biriktirilgan tekislikni kengaytirilgan kompleks tekislik deb ataymiz.

4. $E = \{z: r < |z - a| < R\}$ to‘plam radiuslari r va R, markazi a nuqtada bo‘lgan xalqani ifodalaydi (12.3-rasm). Xususan, $E = \{z: 0 < |z - a| < \varepsilon\}$ to‘plam, a nuqtaning o‘zi kirmagan “xalqani” ifodalab, ba’zan unga a nuqta o‘yib olib tashlangan xalqa deb ataladi.

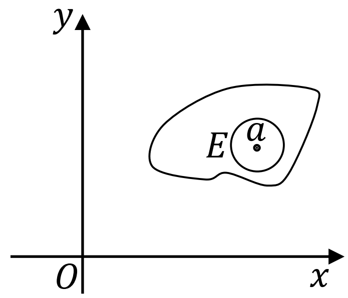
5. $E = \{z: |z - a| + |z - b| = c\}$ shartni qanoatlantiruvchi nuqtalar to‘plami tekislikda, fokuslari a va b nuqtalarda bo‘lgan ellipsdan iborat.

Soha va chegara.

12.2-Ta’rif. z kengaytirilgan tekislikda biror E to‘plam berilgan bo‘lsin. Agar markazi z nuqtada bo‘lgan yetarlicha kichik doiraning barcha nuqtalari E to‘plamga tegishli bo‘lsa, z nuqta E to‘plamning *ichki nuqtasi* deb ataladi, (12.4- rasm).



12.3- rasm



12.4- rasm

Markazi cheksiz uzoqlashgan nuqtada bo‘lgan doira deganda, markazi $z = 0$ nuqtada bo‘lgan ixtiyoriy $|z| = R$ aylananing tashqi qismi tushuniladi.

Agar z_0 nuqtaning ixtiyoriy ε atrofida E to‘plamga tegishli bo‘lgan ham tegishli bo‘lmagan ham nuqtalar yotsa, bu nuqta E to‘plamning *chegaraviy nuqtasi* deb ataladi

E to‘plamning barcha chegaraviy nuqtalari to‘plamiga uning *chegarasi* deyiladi.

12.3-Ta’rif. Agar z nuqta va uning biror ε atrofi berilgan to‘plamga tegishli bo‘lmasa, bu nuqta shu to‘plamning *tashqi nuqtasi* deyiladi.

12.4-Ta’rif. Agar to‘plam faqat ichki nuqtalardan iborat bo‘lsa, u *ochiq to‘plam* deyiladi.

$E = \{z: |z - a| < R\}$ doira va $E = \{z: r < |z - a| < R\}$ xalqa ochiq to‘plamga misol bo‘la oladi.

12.5-Ta’rif. Agar to‘plamning ixtiyoriy ikki nuqtasini shu to‘plamda yotuvchi egri chiziq bilan tutashtirish mumkin bo‘lsa, u *bog‘lamli to‘plam* deyiladi.

12.6-Ta’rif. Kengaytirilgan kompleks tekislikning bog‘lamli ochiq to‘plami *soha* deb ataladi.

12.7-Ta’rif. D soha va uning chegaraviy nuqtalaridan tuzilgan to‘plamga *yopiq to‘plam* deb ataladi va $\overline{D}$ kabi belgilanadi.

$E = \{z: |z - a| \leq R\}$ doira va $E = \{z: r \leq |z - a| \leq R\}$ xalqa yopiq to‘plamga misol bo‘la oladi.

12.2. Kompleks o‘zgaruvchili funksiya, uning limiti va uzluksizligi

Kompleks o‘zgaruvchili funksiyalar. Kompleks sonlarning ikkita z va w tekisliklarini qaraymiz. z tekislikda D soha, w da E soha berilgan bo‘lsin (bu soha ∞ ni ham o‘z ichiga olishi mumkin).

12.8-Ta’rif. Agar D sohaning har bir z nuqtasiga E sohaning bitta yoki bir necha nuqtasi mos qo‘yilgan bo‘lsa, D sohada $w = f(z)$ funksiya berilgan deyiladi (birinchi holda funksiya bir qiymatli, ikkinchi holda ko‘p qiymatli deb ataladi).

D to‘plamga $f(z)$ funksiyaning aniqlanish, E to‘plamga esa qiymatlar sohasi deb ataladi.

$z = x + iy$ va $w = u + iv$ deb olib

$$w = u + iv = f(x + iy) = u(x, y) + iv(x, y)$$

tenglikni hosil qilamiz. Bu yerda $u(x, y) = \mathrm{Re}f(z)$ funksiyaning haqiqiy qismi, $v(x, y) = \mathrm{Im}f(z)$ mavhum qismi deb ataladi.

Shunday qilib kompleks o‘zgaruvchili $w = f(z)$ funksiyaning berilishi haqiqiy o‘zgaruvchili ikkita funksiyaning berilishiga teng kuchli ekan.

12.1-Misol. $w = z^2$ funksiyaning haqiqiy va mavhum qismlarini toping.

►$z = x + iy$ va $w = u + iv$ deb olsak, u holda

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi$$

Demak, $u = x^2 - y^2$ va $v = 2xy$. Bu funksiya bir qiymatli bo‘ladi, chunki har bir z ga uning kvadrati mos qo‘yilmoqda. ◄

Kompleks sondan ildiz chiqarish formulasiga ko‘ra har qanday kompleks sonning kvadrat ildizi ikkita qiymatga ega. Shuning uchun $w = \sqrt{z}$ ikki qiymatli funksiyaga misol bo‘ladi.

$w = f(z)$ funksiyaning aniqlanish sohasi D, qiymatlar sohasi E bo‘lsin. Geometrik jihatdan bu funksiya D sohaning nuqtalarini E sohaning nuqtalariga akslantiradi. Shuning uchun $w = f(z)$ funksiyani akslantirish deb atashimiz mumkin bo‘lib D ga akslantirishning asli, E ga esa tasviri deb ataladi. Yuqoridagi kabi akslantirishlar bir qiymatli va ko‘p qiymatli bo‘lishi mumkin. Bir qiymatli akslantirishda hech qaysi ikki nuqtaning tasvirlari utsma – ust tushmaydi.

Funksiyaning limiti. $f(z)$ funksiya z_0 nuqtaning biror Ω atrofida aniqlangan bo‘lsin (z_0 nuqtaning o‘zida funksiya aniqlanmagan bo‘lishi ham mumkin).

12.9-Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ soni uchun, shunday $\delta > 0$ son mavjud bo'lib, $|z - z_0| < \delta$ tengsizlikni qanoatlantiruvch barcha $z \in \Omega$ nuqtalar uchun

$$|f(z) - W| < \varepsilon$$

tengsizlik o'rinli bo'lsa, W soni $f(z)$ funksiyaning z_0 nuqtadagi limiti deb ataladi.

Bu limit $\lim\limits_{z\to z_0} f(z) = W$ kabi yoziladi, bunda z_0 va W kompleks tekislikning chekli nuqtalari.

Funksiya limitining ta'rifini boshqacha ko'rinishda ham berish mumkin.

12.10-Ta'rif. Agar $\lim\limits_{n\to\infty} z_n = z_0$ tenglikni qanoatlantiruvchi ixtiyoriy $\{z_n\}$ ketma-ketlik uchun $\lim\limits_{n\to\infty} f(z_n) = W$ tenglik o'rinli bo'lsa, W soni $f(z)$ funksiyaning z_0 nuqtadagi limiti deb ataladi.

$W = \infty$ va z_0 chekli bo'lganida funksiyaning limiti quyidagicha ta'riflanadi:

Agar ixtiyoriy $\mathrm{E} > 0$ soni uchun shunday $\delta > 0$ son mavjud bo'lib, $|z - z_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha $z \in \Omega$ nuqtalar uchun

$$|f(z)| > E$$

tengsizlik o'rinli bo'lsa, $A = \infty$ nuqta $f(z)$ funksiyaning z_0 nuqtadagi limiti deb ataladi.

Xuddi shunga o'xshash tarzda cheksiz uzoqlashgan nuqtadagi limit ta'rifini kiritishimiz mumkin (uni kitobxonga mashq tarzida qoldiramiz).

12.1-Teorema. $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ funksiya $z_0 = x_0 + iy_0$ nuqtada $W = a + ib$ limitga ega bo'lishi uchun

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = a, \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = b$$

tengliklarning o'rinli bo'lishi zarur va yetarli.

► $\lim\limits_{n\to\infty} z_n = z_0$ $(z_n = x_n + iy_n)$ bo'lsin. U holda $\lim\limits_{z\to z_0} f(z) = W =$ $= a + ib$ tenglik $\lim\limits_{n\to\infty} f(z_n) = a + ib$ ekanligini anglatadi. Bu tenglik o'rinli bo'lishi uchun mazkur darslikning I qismidagi 1.13-Teoremaga ko'ra $\lim\limits_{n\to\infty} u(x_n, y_n) = a$, $\lim\limits_{n\to\infty} v(x_n, y_n) = b$ tengliklarning bajarilishi zarur va yetarli.◄

Funksiyaning uzluksizligi.

12.11-Ta'rif. $f(z)$ funksiya z_0 chekli nuqtada chekli va $f(z_0)$ qiymatga teng limitga ega, ya'ni

$$\lim_{z\to z_0} f(z) = f(z_0)$$

tenglik o‘rinli bo‘lsa, $f(z)$ funksiya bu *nuqtada uzluksiz* deyiladi.

Cheksiz uzoqlashgan nuqta uchun bu ta’rif quyidagicha ko‘rinish oladi:

12.12-Ta’rif. Agar

$$\lim_{z\to\infty} f(z) = w_0$$

tenglik o‘rinli bo‘lsa, $f(z)$ funksiya $z_0 = \infty$ cheksiz nuqtada uzluksiz deyiladi, bu yerda w_0 chekli son. w_0 ni funksiyaning cheksizlikdagi qiymati deb qabul qilamiz.

12.13-Ta’rif. D sohaning barcha nuqtalarida uzluksiz bo‘lgan $f(z)$ funksiya, shu *sohada uzluksiz* deb ataladi.

Kompleks o‘zgaruvchili funksiya ikkita haqiqiy o‘zgaruvchili funksiyalar orqali ifodalanganligi uchun quyidagi teorema o‘rinli.

12.2-Teorema. $f(z) = u(x,y) + iv(x,y)$ funksiya $z_0 = x_0 + iy_0$ nuqtada uzluksiz bo‘lishi uchun, $u(x,y)$ va $v(x,y)$ funksiyalar (x_0, y_0) nuqtada uzluksiz bo‘lishi zarur va yetarli.

Mazkur teoremaning isboti 12.1-Teoremadan bevosita kelib chiqadi.

Haqiqiy o‘zgaruvchili funksiyalarda bo‘lgani kabi kompleks o‘zgaruvchili funksiyalarda ham quyidagi xossalar o‘rinli.

12.2-Teorema. Agar $f(z)$ va $g(z)$ funksiyalar biror z_0 nuqtada uzluksiz bo‘lsa, u holda

a) $f(z) \pm g(z)$;

b) $f(z) \cdot g(z)$;

c) $\dfrac{f(z)}{g(z)}, (g(z_0) \neq 0)$

funksiyalar ham z_0 nuqtada uzluksiz bo‘ladi.

Agar $f(z)$ funksiya z_0 nuqtada, $F(w)$ funksiya esa $w_0 = f(z_0)$ nuqtada uzluksiz bo‘lsa, u holda $F\big(f(z)\big)$ murakkab funksiya z_0 nuqtada uzluksiz bo‘ladi.

Kompleks o‘zgaruvchili asosiy elementar funksiyalar.

1. Chiziqli funksiya $w = az + b,\ \ a \neq 0$
2. Kasr chiziqli funksiya $w = (az + b)/(cz + d),\ \ c \neq 0, ad - bc \neq 0$
3. Ko‘rsatgichli funksiya $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

Bu funksiya quyidagi xossalarga ega:

a) $e^{z_1+z_2} = e^{z_1}e^{z_2}$;

b) $e^{z_1-z_2}=e^{z_1}/e^{z_2}$;

c) $e^{z+2\pi ki}=e^z$, ya'ni e^z funksiya davriy bo'lib, uning davri $2\pi i$ ga teng.

4. Trigonometrik funksiyalar ko'rsatgichli funksiya yordamida quyidagicha aniqlanadi:

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}, \cos z=\frac{e^{iz}+e^{-iz}}{2},$$

$$\operatorname{tg} z=\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}, \operatorname{ctg} z=\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$$

yoki

$$\operatorname{tg} z=\frac{\sin z}{\cos z}, \operatorname{ctg} z=\frac{\cos z}{\sin z}$$

Shuni ta'kidlash joizki, bu funksiyalarning argumentlari xususan haqiqiy son bo'lganida ular haqiqiy ma'nodagi trigonometrik funksiyalar bilan ustma-ust tushadi. Bundan tashqari ko'pchilik trigonometrik formulalar ular uchun saqlanib qoladi.

5. Giperbolik funksiyalar haqiqiy o'zgaruvchili funksiyalardagi bo'lgani kabi aniqlanadi

$$\mathrm{sh}z=\frac{e^z-e^{-z}}{2}, \mathrm{ch}z=\frac{e^z+e^{-z}}{2}, \mathrm{th}z=\frac{\mathrm{sh}z}{\mathrm{ch}z}, \mathrm{cth}z=\frac{\mathrm{ch}z}{\mathrm{sh}z}$$

Aniqlanishidan ko'rinib turibdiki trigonometrik va giperbolik funksiyalar o'zaro bog'liq. Ularning ta'rifidan bevosita

$$\begin{array}{ll} \sin z=-i\,\mathrm{sh}iz & \mathrm{sh}z=-i\sin iz \\ \cos z=\mathrm{ch}\,iz & \mathrm{ch}z=\cos iz \\ \mathrm{tg}z=-i\,\mathrm{th}\,iz & \mathrm{th}z=-i\,\mathrm{th}\,iz \\ \mathrm{ctg}z=i\,\mathrm{cth}\,iz & \mathrm{cth}z=i\,\mathrm{ctg}\,iz \end{array}$$

tengliklar kelib chiqadi. Bu formulalardan ko'pincha, trigonometrik funksiyalarning kompleks argumentdagi qiymatlarini hisoblashda foydalaniladi. Ma'lumki, trigonometrik funksiyalar uchun ko'plab ayniyatlar bajariladi. Yuqoridagi bog'lanishlarni e'tiborga olgan holda giperbolik funksiyalar uchun ham ba'zi ayniyatlarni isbotlash mumkin.

6. Logarifmik funksiya $w=\mathrm{Ln}z, z\neq 0$

Kompleks sonning $\mathrm{Ln}z$ logarifmini haqiqiy sonlardagi kabi $w=e^z$ tenglikda ma'lum w ga ko'ra z ni aniqlashdan hosil qilamiz. $w=e^z$ funksiya davriy bo'lganligi uchun unga teskari bo'lgan $w=\mathrm{Ln}z$ funksiya cheksiz ko'p qiymatli bo'ladi va uni

$$\mathrm{Ln}z = \ln|z| + i\mathrm{Arg}z = \ln|z| + i\arg z + 2k\pi i, \quad k \in Z$$
kabi aniqlaymiz. Bu yerda $\ln z = \ln|z| + i\arg z$ ga funksiyaning bosh qiymati deb ataladi. Yuqoridagi kabi aniqlangan logarifmik funksiya haqiqiy o‘zgaruvchilardagi kabi xossalarga ega.

7. Teskari trigonometrik funksiyalarni
$$\mathrm{Arcsin}z = -i\mathrm{Ln}(iz + \sqrt{1 - z^2})$$
$$\mathrm{Arccos}z = -i\mathrm{Ln}(z + \sqrt{z^2 - 1})$$
$$\mathrm{Arctg}z = -\frac{i}{2}\mathrm{Ln}\frac{1 + iz}{1 - iz}$$
$$\mathrm{Arcctg}z = -\frac{i}{2}\mathrm{Ln}\frac{z + i}{z - i}$$
formulalar bilan aniqlaymiz. Bu funksiyalarning barchasi ko‘p qiymatli bo‘lib, ularning bosh qiymati mos ravishda logarifmning bosh qiymatiga mos olinadi.

8. Umumiy darajali funksiya $w = z^a$.

Bu yerda w ning qiymati $z^a = e^{a\mathrm{Ln}z}$ tenglikdan aniqlanadi, uning bosh qiymati esa $z^a = e^{a\ln z}$ kabi olinadi.

9. Umumiy ko‘rsatgichli funksiya $w = a^z, a \neq 0$

Bu holda ham yuqoridagi kabi $a^z = e^{z\mathrm{Ln}a}$ tenglikdan foydalanamiz.

12.2-Misol. $\sin(\pi + i\ln 3)$ ni hisoblang.

►$\sin z = -i\,\mathrm{sh}\,iz$ ekanligidan
$$\sin(\pi + i\ln 3) = -i\mathrm{sh}(\pi i - \ln 3) = -i\frac{e^{\pi i - \ln 3} - e^{-\pi i + \ln 3}}{2} =$$
$$= -\frac{i}{2}\left(e^{\pi i}e^{-\ln 3} - e^{-\pi i}e^{\ln 3}\right) = -\frac{i}{2}\left(\frac{1}{3}e^{\pi i} - 3e^{-\pi i}\right) =$$
$$= -\frac{i}{2}\left(-\frac{1}{3} + 3\right) = -\frac{4i}{3}.$$
qiymatni hosil qilamiz. ◄

12.3-Misol. i^i ni hisoblang.

►Umumiy darajali funksiyaning ta’rifiga ko‘ra
$$i^i = e^{i\mathrm{Ln}i} = e^{i(\ln 1 + i\arg i + 2k\pi i)} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)}$$
$k = 0$ da bosh qiymat $i^i = e^{-\frac{\pi}{2}}$ ni hosil qilamiz. ◄

12.3. Kompleks o‘zgaruvchili funksiyalarni differensiallash

Kompleks o‘zgaruvchili funksiyaning hosila va differensiali ta’rifi haqiqiy o‘zgaruvchili funksiyalarning mos ta’rifi bilan aynan ustma-ust tushadi. Shuning uchun differensial hisobning barcha teorema va formulalari kompleks o‘zgaruvchili funksiyalarga ham targ‘ib etiladi.

Shu bilan birga kompleks o‘zgaruvchili differensiallanuvchi funksiyalar ko‘pchilik qo‘shimcha xossalarga ham ega. Ularning paydo bo‘lishi sababi, kompleks o‘zgaruvchili funksiya differensiallanuvchiligi sharti haqiqiy o‘zgaruvchili funksiya hosilasi mavjudligi shartidan ko‘ra birmuncha chegaralanganligidadir.

Kompleks o‘zgaruvchili funksiya hosilasi. $w = f(z)$ funksiya berilgan bo‘lsin. $z = x + iy$ argumentga $\Delta z = \Delta x + i\Delta y$ orttirma beramiz va funksiyaning $\Delta w = f(z + \Delta z) - f(z)$ orttirmasini hisoblaymiz.

12.13-Ta’rif. Agar Δz orttirma ixtiyoriy qonun bilan nolga intilganida $\Delta w/\Delta z$ ifodaning limiti mavjud bo‘lsa, bu limitga $f(z)$ funksiyaning z nuqtadagi hosilasi deb ataladi va

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = f'(z) = \frac{dw}{dz} = \frac{df}{dz} \tag{12.1}$$

kabi belgilanadi, $f(z)$ funksiya esa bu nuqtada differensiallanuvchi deyiladi.

12.4-Misol. $f(z) = C$ $(C - const)$ funksiya butun $\mathbb{C}$ kompleks tekislikda differensiallanuvchi va $(C)' = 0$.

► $\lim\limits_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{C - C}{\Delta z} = 0$, ya’ni $(C)' = 0$.◄

12.5-Misol. $f(z) = z^n$ $(n \in \mathbb{N})$ funksiya butun $\mathbb{C}$ kompleks tekislikda differensiallanuvchi va $(z^n)' = nz^{n-1}$.

$$► \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{z^n + C_n^1 z^{n-1}\Delta z + C_n^2 z^{n-2}(\Delta z)^2 + \cdots C_n^n (\Delta z)^n - z^n}{\Delta z} =$$

$$\lim_{\Delta z \to 0} (C_n^1 z^{n-1} + C_n^2 z^{n-2}\Delta z + \cdots + C_n^n (\Delta z)^{n-1}) = C_n^1 z^{n-1} = nz^{n-1},$$

ya’ni $(z^n)' = nz^{n-1}$.◄

$\Delta w/\Delta z$ nisbatning Δz nolga ixtiyoriy tarzda intilganida limitga ega bo‘lishi sharti, haqiqiy o‘zgaruvchili $\varphi(x)$ funksiya uchun qo‘yilgan shartdan kuchliroqdir.

Bu shart $z+\Delta z$ ning z ga ixtiyoriy yo‘l bilan, jumladan z ga keluvchi ixtiyoriy nur bo‘ylab yaqinlashganida ham $\Delta w/\Delta z$ nisbatning limiti mavjud va ularning barchasi teng bo‘lishi kerakligini anglatadi. Biz bu yaqinlashishlarning ikkita maxsus holini qarab chiqamiz.

Koshi- Riman shartlari. $w=f(z)=u(x,y)+iv(x,y)$ funksiya berilgan bo‘lsin. $\Delta z=\Delta x+i\Delta y$ orttirma uchun funksiya orttirmasini quyidagicha yozamiz

$$\Delta w=f(z+\Delta z)-f(z)=[u(x+\Delta x,y+\Delta y)-u(x,y)]+$$
$$+i[v(x+\Delta x,y+\Delta y)-v(x,y)]=\Delta u+i\Delta v$$

bu yerda $\Delta u=u(x+\Delta x,y+\Delta y)-u(x,y)$ va $\Delta v=v(x+\Delta x,y+\Delta y)-v(x,y)$

Hosila ta’rifiga ko‘ra

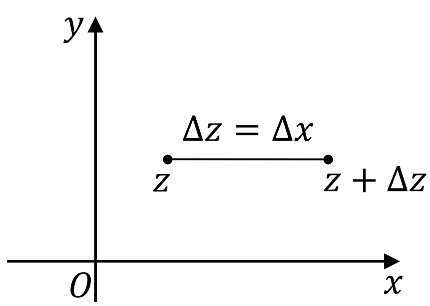
$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y} \qquad (12.2)$$

$f(z)$ funksiya z nuqtada hosilaga ega bo‘lsin, u holda $f'(z)=\lim\limits_{\Delta z\to 0}\frac{\Delta w}{\Delta z}$ limit mavjud va u $\Delta z=\Delta x+i\Delta y$ orttirmaning nolga intilish turiga bog‘liq emas. Shuni hisobga olib biz quyidagi xususiy hollarni qaraymiz:

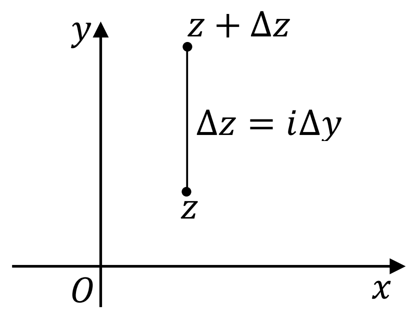
1) $\Delta z=\Delta x$ bo‘lsin, ya’ni $z+\Delta z$ ning z nuqtaga yaqinlashishi Ox o‘qiga parallel (12.5 rasm). U holda

$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta x\to 0}\frac{\Delta u+i\Delta v}{\Delta x}=$$
$$=\lim_{\Delta x\to 0}\left(\frac{\Delta u}{\Delta x}+i\frac{\Delta v}{\Delta x}\right)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \qquad (12.3)$$

tenglikni hosil qilamiz.



12.5- rasm



12.6-rasm

2) $\Delta z=i\Delta y$, ya’ni $z+\Delta z$ ning z nuqtaga yaqinlashishi Oy o‘qiga parallel bo‘lsin (12.6-rasm). Bu holda yuqoridagi kabi amal bajarib

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} =$$

$$= \lim_{\Delta y \to 0} \left(\frac{\Delta u}{i\Delta y} + \frac{\Delta v}{\Delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (12.4)$$

tenglikga ega bo‘lamiz.

$\Delta w/\Delta z$ ifodaning $\Delta z \to 0$ dagi limiti, Δz ning nolga intilish turiga bog‘liq bo‘lmaganligi uchun (12.3) va (12.4) limitlar o‘zaro teng, ya’ni

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Bu yerdan

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad (12.5)$$

(12.5) tengliklarga *Koshi- Riman shartlari* deb ataladi.

Ular $f(z)$ funksiya differensiallanuvchi bo‘lgan barcha nuqtalarda bajarilishi kerak. Demak funksiya differensiallanuvchi bo‘lishi uchun bu funksiya Koshi- Riman shartlarini qanoatlantirishi zarur ekan. $u(x,y)$ va $v(x,y)$ funksiyalarning differensiallanuvchiligi sharti talab etilganida, bu shartlar yetarli ham bo‘lar ekan.

Haqiqatan ham, $u(x,y)$ va $v(x,y)$ funksiyalar differensiallanuvchi bo‘lsin. U holda (1 qism, 9.1 va 9.2-Teorema) $u(x,y)$ va $v(x,y)$ funksiyalar uchun

$$\Delta u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \alpha_1(\Delta x, \Delta y)\rho$$

$$\Delta v(x,y) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \alpha_2(\Delta x, \Delta y)\rho$$

tengliklarni hosil qilamiz, bu yerda $\rho = \sqrt{\Delta x^2 + \Delta y^2}$ va α_1 hamda α_2 miqdorlar ρ miqdorga nisbatan cheksiz kichik.

Yuqoridagilarga asosan ixtiyoriy $\Delta z = \Delta x + i\Delta y$ uchun

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \frac{\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + \rho(\alpha_1 + i\alpha_2)}{\Delta x + i\Delta y}$$

Bu tenglikda Koshi- Riman shartidan foydalanib $\frac{\partial v}{\partial y}$ hosilani $\frac{\partial u}{\partial x}$ bilan, $\frac{\partial u}{\partial y}$ hosilani esa $-\frac{\partial v}{\partial x}$ bilan almashtiramiz va

$$\frac{\Delta w}{\Delta z}=\frac{\frac{\partial u}{\partial x}\Delta x-\frac{\partial v}{\partial x}\Delta y+i\left(\frac{\partial v}{\partial x}\Delta x+\frac{\partial u}{\partial x}\Delta y\right)+\rho(\alpha_1+i\alpha_2)}{\Delta x+i\Delta y}=$$

$$\frac{\frac{\partial u}{\partial x}(\Delta x+i\Delta y)+i\frac{\partial v}{\partial x}(\Delta x+i\Delta y)+\rho(\alpha_1+i\alpha_2)}{\Delta x+i\Delta y}=$$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\frac{\rho(\alpha_1+i\alpha_2)}{\Delta x+i\Delta y} \qquad (12.6)$$

tenglikga ega bo‘lamiz. $\rho=\sqrt{\Delta x^2+\Delta y^2}=|\Delta z|$ ekanligidan, $\left|\frac{\rho(\alpha_1+i\alpha_2)}{\Delta x+i\Delta y}\right|=|\alpha_1+i\alpha_2|$. α_1 va α_2 miqdorlar ρ miqdorga nisbatan cheksiz kichik bo‘lganligi uchun $\rho\to 0$ da $|\alpha_1+i\alpha_2|\to 0$. O‘z navbatida $\Delta z\to 0$ da (12.6) ga asosan

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},$$

ya’ni $f(z)$ funksiya differensiallanuvchi ekan.

Shunday qilib quyidagi teoremani isbot qildik. ►

12.3-Teorema. $f(z)=u(x,y)+iv(x,y)$ funksiya $z=x+iy$ nuqtada differensiallanuvchi bo‘lishi uchun

1. $u(x,y)$ va $v(x,y)$ funksiyalar (x,y) nuqtada differensiallanuvchi;
2. (x,y) nuqtada Koshi- Riman shartlari bajarilishi zarur va etarli.

Koshi- Riman shartlariga ko‘ra hosilani

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

kabi formulalar bilan hisoblash mumkin.

12.14-Ta’rif. Agar funksiya nafaqat berilgan nuqtada, balki uning biror atrofida ham differensiallanuvchi bo‘lsa, unga shu nuqtada *analitik* deyiladi.

12.15-Ta’rif. Biror sohaning barcha nuqtalarida analitik bo‘lgan funksiya shu sohada analitik yoki *golomorf* deb ataladi.

12.16-Ta’rif. Bir qiymatli $f(z)$ funksiyaning analitiklik nuqtalari uning *to‘g‘ri nuqtalari* deb aytiladi, qolgan barcha nuqtalari uning maxsus nuqtalari deyiladi.

Xususan, funksiya aniqlanmagan nuqtalar ham uning maxsus nuqtasi bo‘ladi. Amaliy masalalarda funksiyani analitiklikka tekshirish uchun, uning Koshi- Riman shartlarini qanoatlantirishini tekshirish etarli bo‘lar ekan.

Shuni yana bir marta ta’kidashimiz joizki, haqiqiy o‘zgaruvchili funksiyalar uchun o‘rinli bo‘lgan barcha teoremalar kompleks o‘zgaruvchili funksiyalar uchun ham o‘rinlidir. Xususan, bizga ma’lum bo‘lgan hosila olish qoidalari va hosilalar jadvali kompleks o‘zgaruvchili funksiyalar uchun ham o‘rinli. Faqatgina hosila olish mumkin bo‘lishi uchun Koshi- Riman shartlari bajarilishi lozim.

12.6-Misol. $w = z^2$ funksiyani analitiklikka tekshiring.

►$w = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$

Demak $u = x^2 - y^2$ va $v = 2xy$. Bu funksiyalar Koshi- Riman shartlarini qanoatlantiradi, chunki

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Bu tengliklar tekislikning barcha nuqtalarida o‘rinli bo‘lganligi uchun, berilgan funksiya butun tekislikda analitik bo‘ladi hamda

$$w' = (z^2)' = 2z. ◄$$

12.7-Misol. $w = z\bar{z}$ funksiyani analitiklikka tekshiring.

► Funksiyaning haqiqiy va mavhum qismlarini ajratamiz

$$w = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

$$u = x^2 + y^2, \ v = 0$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y, \qquad \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0$$

Ko‘rinib turibdiki, Koshi- Riman shartlari faqatgina bir nuqta $x = 0$ va $y = 0$ bo‘lgandagina bajariladi. Shuning uchun berilgan funksiya faqat bitta $z = 0$ nuqtada diffrerentsiallanuvchi bo‘lib, $f'(z) = 0$. Shu bilan birga birorta ham nuqtada analitik emas. ◄

Qo‘shma garmonik funksiyalar. $f(z) = u(x, y) + iv(x, y)$ funksiya biror D sohada differensiallanuvchi, u va v funksiyalar esa ikkinchi tartibgacha uzluksiz xususiy hosilalarga ega bo‘lsin. (12.5) Koshi-Riman shartlarining birinchisini x bo‘yicha, ikkinchisini y bo‘yicha differensiallab

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}, \quad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}$$

tengliklarni hosil qilamiz. Bu tengliklarni $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$ ekanligini e’tiborga olgan holda ayirsak

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (12.7)$$

tenglikga ega bo‘lamiz. Xuddi yuqoridagi kabi amallarni bajarsak, $v(x,y)$ funksiya ham

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

shartni qanoatlantirishiga ishonch hosil qilamiz

(12.7) tenglikga Laplas tenglamasi deb ataladi. Laplas tenglamasini qanoatlantiruvchi funksiyaga esa *garmonik funksiya* deyiladi.

$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ ifoda Laplas operatori deb ataladi. Bunga ko‘ra (12.7) tenglikni

$$\Delta u=0$$

deb yozishimiz mumkin.

Biror D sohada analitik bo‘lgan funksiya ixtiyoriy tartibdagi hosilaga ega bo‘lganligi uchun, uning haqiqiy va mavhum qismlari ham ixtiyoriy tartibdagi xususiy hosilalarga ega. Demak, yuqorida keltirib chiqarilganiga ko‘ra, ixtiyoriy analitik funksiyaning haqiqiy va mavhum qismlari garmonik funksiyalardan iborat bo‘lar ekan.

12.17-Ta’rif. Koshi-Riman shartlari bilan bog‘langan $u(x,y)$ va $v(x,y)$ garmonik funksiyalar o‘zaro *qo‘shma garmonik* funksiyalar deb ataladi.

Umuman olganda $u(x,y)$ va $v(x,y)$ funksiyalarning garmonik ekanligidan $u(x,y)+iv(x,y)$ funksiyaning analitik bo‘lishi kelib chiqmaydi. Bu funksiya analitik bo‘lishi uchun qo‘shimcha ravishda Koshi-Riman shartlarining bajarilishi talab etiladi. Qo‘shma garmonik funksiyalar uchun quyidagi teorema o‘rinli.

12.4-Teorema. Bir bog‘lamli D sohada garmonik bo‘lgan ixtiyoriy $u(x,y)$ funksiya uchun, u bilan qo‘shma garmonik bo‘lgan funksiyani ixtiyoriy o‘zgarmas qo‘shiluvchi aniqligida topish mumkin.

Bu teoremaning ma’nosi shundan iboratki, analitik funksiyaning haqiqiy yoki mavhum qismini bilgan holda, ixtiyoriy o‘zgarmas qo‘shiluvchi aniqligida funksiyaning o‘zini tiklash mumkin.

12.8-Misol. Haqiqiy qismi $u(x,y)=\ln(x^2+y^2)$ tenglik bilan aniqlanuvchi analitik funksiyani toping.

► Qidirilayotgan funksiya analitik bo‘lganligi uchun, u funksiya Koshi – Riman shartlarini qanoatlantiradi. Berilgan $u(x,y)=\ln(x^2+y^2)$ funksiya uchun

$$\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2},\qquad \frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}$$

Koshi-Rimanning ikkinchi shartidan

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\ \text{ yoki }\ \frac{\partial v}{\partial x}=-\frac{2y}{x^2+y^2}$$

Bu tenglikni x bo‘yicha integrallasak

$$v(x,y)=-\int\frac{2y}{x^2+y^2}dx+c(y)=-2\text{arctg}\frac{x}{y}+c(y)$$

Oxirgi funksiyani y bo‘yicha differensiallaymiz

$$\frac{\partial v}{\partial y}=\frac{2x}{x^2+y^2}+c'(y)$$

va Koshi-Rimanning birinchi sharti, ya’ni $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ tenglikdan

$$\frac{2x}{x^2+y^2}=\frac{2x}{x^2+y^2}+c'(y)$$

ni hosil qilamiz. Bu yerdan $c'(y)=0$ yoki $c(y)=c$. Demak

$$v(x,y)=-2\text{arctg}\frac{x}{y}+c$$

Shunday qilib

$f(z)=\ln(x^2+y^2)-2i\text{arctg}\frac{x}{y}+c.$

Agar $\ln(x^2+y^2)=2\ln|z|$, $\text{arctg}\frac{x}{y}=\frac{\pi}{2}-\text{arctg}\frac{y}{x}=\frac{\pi}{2}-\arg z$

ekanligini e’tiborga olsak, logarifmning ta’rifidan

$$f(z)=2\ln(x^2+y^2)-2i\text{arctg}\frac{x}{y}+c=2\ln z+c_1$$

tenglik hosil bo‘ladi. ◄

Hosila moduli va argumentining geometrik ma’nosi. $f(z)$ funksiya biror D sohada analitik bo‘lsin. z_0 bu sohada yotuvchi qandaydir nuqta bo‘lib, $f'(z_0)\neq 0$ shart bajarilsin. Bu funksiya z tekislikni w tekislikka o‘tkazib, z_0 nuqtani esa w_0 ga akslantirsin.

γ chiziq z_0 nuqta orqali, uning aksi bo‘lgan Γ chiziq esa w_0 nuqta orqali o‘tsin. $z_0+\Delta z$ nuqta γ chiziq bo‘ylab harakatlanib z_0 ga yaqinlashsin, unga mos bo‘lgan $w_0+\Delta w$ nuqta esa Γ chiziqda w_0 ga yaqinlashsin.

Ta’rifga ko‘ra

$$|f'(z_0)| = \lim_{\Delta z \to 0} \left|\frac{\Delta w}{\Delta z}\right| = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|}$$

bu yerda $|\Delta z|$ soni z_0 va $z_0 + \Delta z$ nuqtalar orasidagi masofaga teng, xuddi shu kabi $|\Delta w|$ ham w_0 va $w_0 + \Delta w$ nuqtalar orasidagi masofaga teng. Demak

$$|f'(z_0)| = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|}$$

tenglikda limitning qiymati $z_0 + \Delta z$ nuqtaning z_0 ga qanday yaqinlashishiga bog‘liq bo‘lmasdan, ikki cheksiz kichik masofalar nisbatlariga bog‘liq ekan. Bu munosabatning geometrik ma’nosi shundaki, funksiya analitik bo‘lgan nuqtada barcha yo‘nalishlarda masofalar cho‘zilishi bir xil bo‘lar ekan. Shunday qilib, hosila modulining geometrik ma’nosi quyidagicha:

$|f'(z_0)|$ *soni* $w = f(z)$ *akslantirishning* z_0 *nuqtadagi cho‘zilish (siqilish) koeffitsientini aniqlaydi*.

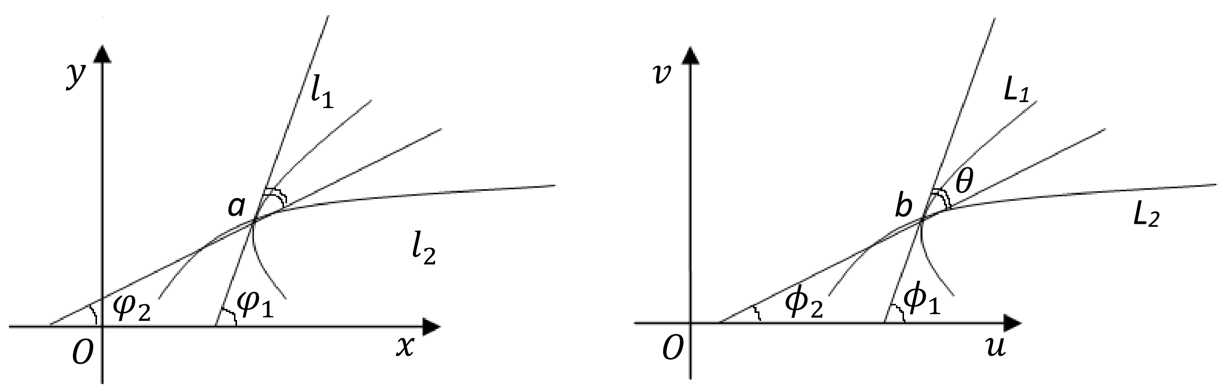
Unga *cho‘zilishlar o‘zgarmasligi xossasi* deb ataladi.

Hosilaning argumenti uchun

$$arg f'(z_0) = \lim_{\Delta z \to 0} arg \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} (arg\Delta w - arg\Delta z) =$$
$$= \lim_{\Delta w \to 0} arg\Delta w - \lim_{\Delta z \to 0} arg\Delta z = \Phi - \varphi$$

bu yerda Φ va φ mos ravishda Γ hamda γ chiziqlarning w_0 va z_0 nuqtalardagi urinmalarining haqiqiy o‘q musbat yo‘nalishi bilan hosil qilgan burchaklari. Agar $\Phi = \varphi + arg f'(z_0)$ deb yozib olsak, hosila argumentining geometrik ma’nosi quyidagicha tavsiflanadi:

$\arg f'(z_0)$ *burchak* $w = f(z)$ *akslantirishda* z_0 *nuqtada ixtiyoriy* γ *chiziq urinmasining* Γ *chiziq urinmasiga burilish kerak bo‘lgan burchagiga teng. Agar* $\arg f'(z_0) > 0$ *bo‘lsa, burilish soat strelkasi yo‘nalishiga qarama qarshi,* $\arg f'(z_0) < 0$ *bo‘lsa soat strelkasi yo‘nalishi bo‘ylab.*



12.7-rasm

$w = f(z)$ analitik funksiya berilgan bo‘lsin. a nuqtada θ burchak ostida kesishuvchi ikkita silliq l_1 va l_2 (θ bu chiziqlarga o‘tkazilgan urinmalar orasidagi $\theta = \varphi_2 - \varphi_1$ burchak) chiziqlarning akslari L_1 va L_2 ham silliq chiziqlar bo‘lib, $b = f(a)$ shart bajarilsin (12.7- rasm).

Funksiya analitik, shuning uchun $\phi_1 = \varphi_1 + \arg f'(a)$, $\phi_2 = \varphi_2 + \arg f'(a)$.

Demak $\phi_1 - \phi_2 = \varphi_1 - \varphi_2$. Shartga ko‘ra $\varphi_1 - \varphi_2 = \theta$, u holda $\phi_1 - \phi_2 = \theta$. Bu esa $b = f(a)$ nuqtadagi L_1 va L_2 chiziqlar orasidagi burchak.

Shunday qilib analitik funksiya bajaradigan akslantirishda l_1 va l_2 chiziqlar orasidagi burchak θ bo‘lsa, ularning aksi bo‘lgan L_1 va L_2 chiziqlar orasidagi burchak ham θ bo‘lar ekan. Unga burchaklar *saqlanish xossasi* deb ataladi.

12.18-Ta’rif. Burchaklar saqlanishi va cho‘zilishlar o‘zgarmasligi xossasiga ega bo‘lgan akslantirishga *konform akslantirish* deb ataladi.

Analitik funksiya bajaradigan akslantirish, agar $f'(z_0) \neq 0$ bo‘lsa konform bo‘ladi.

12.9-Misol. $w = z^2$ funksiya bajaradigan akslantirishda $z_0 = 1 + i$ nuqtadagi cho‘zilish koeffitsienti va burilish burchagini toping.

►$w' = 2z = 2x + 2iy$

Demak $w'(z_0) = 2 + 2i$

$$|w'(z_0)| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\arg w'(z_0) = \mathrm{arctg} 1 = \frac{\pi}{4}$$

Shunday qilib $1 + i$ nuqtadan o‘tuvchi silliq chiziqlar $w = z^2$ akslantirishda bir xil $2\sqrt{2}$ cho‘zilishga ega. Bu chiziqlarning akslari $w_0 = (1 + i)^2 = 2i$ nuqtada bir xil $\pi/4$ burchakka buriladi. ◄

12.4. Kompleks o‘zgaruvchili funksiyalarni integrallash

Kompleks o‘zgaruvchili funksiyalarning integrali ta’rifi. z kompleks tekislikda yopiq yoki yopiq bo‘lmagan silliq yoki bo‘lakli silliq C yoy berilgan bo‘lsin. Chiziqning boshlang‘ich nuqtasini z_0, oxirgi nuqtasini esa z (agar chiziq yopiq bo‘lsa $z_0 = z$) deb olamiz. z_0 ni boshlang‘ich, z ni oxirgi nuqta deb tanlab, C chiziqda musbat yo‘nalishni tanlab oldik. $f(z)$ funksiyani C chiziqning barcha nuqtalarida uzluksiz deb faraz qilaylik.

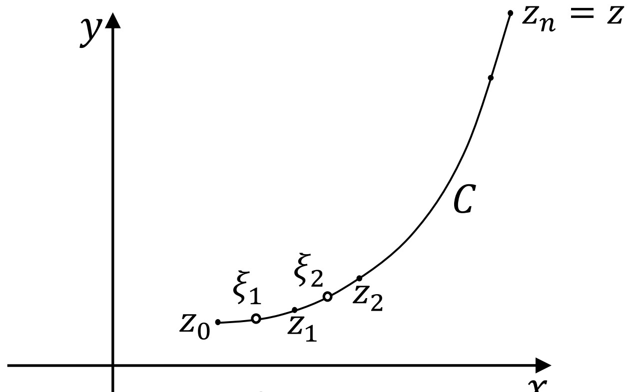
C yoyni ixtiyoriy usulda n ta "elementar" yoylarga ajratamiz va bo'linish nuqtalarini z_0 dan z ga bo'lgan yo'nalishda z_k kabi tartib raqamlarini qo'yib chiqamiz, bu yerda $z_n = z$.

$z_1 - z_0 = \Delta z_1$, $z_2 - z_1 = \Delta z_2$,…, $z_n - z_{n-1} = \Delta z_n$ belgilashlar kiritamiz.

Δz_k soni z_{k-1} nuqtadan z_k nuqtaga qarab yo'nalgan vektor bilan tasvirlanadi, $|\Delta z_k|$ esa bu vektoring uzunligi. Har bir elementar yoyda ixtiyoriy tarzda bittadan $\xi_1, \xi_2, \dots, \xi_n$ nuqta tanlaymiz. (12.8-rasm) Yuqoridagi bo'linishga asosan quyidagi yig'indini tuzamiz

$$f(\xi_1)\Delta z_1 + f(\xi_2)\Delta z_2 + \dots + f(\xi_n)\Delta z_n \qquad (12.8)$$

va $\lambda = \max\limits_k |\Delta z_k|$ kabi belgilash kiritamiz.



12.8- rasm

12.19-Ta'rif. $\lambda \to 0$ ($n \to \infty$) da (12.8) yig'indining limiti mavjud bo'lib, bu limitning qiymati chiziqning bo'linish turiga va nuqtalarning tanlanish usuliga bog'liq bo'lmasa, limitning qiymatiga $f(z)$ funksiyadan C yoy bo'yicha olingan integral deb ataladi va

$$\int\limits_C f(z)dz = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta z_k \qquad (12.9)$$

kabi belgilanadi.

Agar C integrallash chizig'i yopiq, ya'ni C yopiq konturdan iborat bo'lsa, kontur bo'yicha integrallash yo'nalishiga qarab

$$\oint\limits_C f(z)dz, \oint\limits_C f(z)dz$$

belgilardan biri qo'llaniladi.

Shuni ta'kidlab o'tishimiz lozimki, egri chiziqli integrallar kabi bu integral ham integrallash chizig'i silliq va integrallanuvchi funksiya uzluksiz bo'lganida har doim mavjud.

Kompleks o‘zgaruvchili funksiyalar integrallarining xossalari. Kompleks o‘zgaruvchili funksiya integrali ta’rifidan ko‘rinib turibdiki, uni haqiqiy o‘zgaruvchili funksiyaning egri chiziqli integraliga olib kelish mumkin. Haqiqatan, haqiqiy o‘zgaruvchilarga o‘tsak,

$$z = x + iy, \;\; f(z) = u(x, y) + iv(x, y)$$

tengliklarni e’tiborga olgan holda

$$\int_C f(z)dz = \int_C [u(x, y) + iv(x, y)](dx + idy)$$

yoki

$$\int_C f(z)dz = \int_C u(x, y)dx - v(x, y)dy + i\int_C v(x, y)dx + u(x, y)dy \qquad (12.10)$$

tenglikni hosil qilamiz.

Demak, kompleks o‘zgaruvchili funksiyaning integrali ikkita egri chiziqli integrallar yig‘indisiga kelar ekan. Shuning uchun egri chiziqli integrallardan kelib chiqadigan bir nechta xossalarni isbotsiz keltiramiz.

1. $\int_C [f_1(z) \pm f_2(z)]dz = \int_C f_1(z)dz + \int_C f_2(z)dz,$

2. $\int_C kf(z)dz = k\int_C f(z)dz,$

bu yerda $k - const$;

3. Agar $\bar{C}$ – geometrik jihatdan C bilan ustma ust tushib, lekin unda yo‘nalish qarama qarshi bo‘lsa, u holda

$$\int_{\bar{C}} = -\int_C f(z)dz;$$

4. Agar C yoy $C_1, C_2, ..., C_n$ yoylardan tashkil topgan bo‘lsa, u holda

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz;$$

5. $\int_C dz = z - z_0;$

6. Agar C yoyning barcha nuqtalarida $|f(z)| \leq M$ tengsizlik o‘rinli bo‘lib, C yoyning uzunligi l ga teng bo‘lsa, u holda

$$\left|\int_C f(z)dz\right| \leq Ml$$

Bu xossalar integralning ta’rifidan bevosita kelib chiqadi, shuning uchun ularning isbotiga to‘xtalib o‘tirmaymiz.

Kompleks o‘zgaruvchili funksiya integralini hisoblash. Agar C chiziq $x = x(t), y = y(t)$ parametrik tenglamasi bilan berilgan bo‘lib, chiziqning boshlang‘ich va oxirgi nuqtalari $t = \alpha, t = \beta$ parametrlarga mos kelsa, u holda

$$\int_C f(z)dz = \int_\alpha^\beta f\big(z(t)\big)z'(t)dt, \qquad (12.11)$$

bu yerda $z(t) = x(t) + iy(t)$.

Xususan, agar chiziq $y = y(x), a < x < b$ ko‘rinishdagi tenglama bilan berilgan bo‘lsa, u holda (12.11) formula quyidagicha bo‘ladi

$$\int_C f(z)dz = \int_a^b f\big(x + iy(x)\big)(1 + iy'(x))dx$$

12.10-Misol. Ushbu $\int_C \bar{z}dz$ integral xisoblansin, bu yerda $C -$chiziq $z = 0$ va $z = 1 + i$ nuqtalarni tutashtirib turuvchi $y = x$ to‘g‘ri chiziq kesmasidan iborat.

►Agar bu integralda haqiqiy o‘zgaruvchilarga o‘tsak

$$\int_C \bar{z}dz = \int_C (x - iy)d(x + iy) = (1 - i^2)\int_0^1 xdx = 1$$

hosil bo‘ladi. ◄

12.20-Ta’rif. Agar

$$\Phi'(z) = f(z)$$

tenglik o‘rinli bo‘lsa, $\Phi(z)$ funksiya $f(z)$ funksiyaning boshlang‘ich funksiyasi deb ataladi

Haqiqiy o‘zgaruvchili funksiyalar uchun o‘rinli bo‘lgan Nyuton-Leybnits formulasini kompleks o‘zgaruvchili funksiyalarga ham ko‘chirishimiz mumkin.

12.5-Teorema. Agar $f(z)$ funksiya z_0 va z_1 nuqtalarni o‘z ichiga oluvchi *D* sohada analitik bo‘lsa, u holda Nyuton-Leybnis formulasi o‘rinli:

$$\int_{z_0}^{z_1} f(z)dz = \Phi(z_1) - \Phi(z_0), \qquad (12.12)$$

bu yerda $\Phi(z)$ funksiya $f(z)$ ning biror boshlang‘ich funksiyasi.

Endi biz ko‘pchilik integrallarni hisoblashda muhim bo‘lgan ikkita xossani keltiramiz.

12.6-Teorema. $f(z)$ va $g(z)$ biror bir bog‘lamli *D* sohada analitik funksiyalar, z_0 va z_1 bu sohaning ixtiyoriy nuqtalari bo‘lsa, u holda quyidagi bo‘laklab integrallash formulasi o‘rinli

$$\int_{z_0}^{z_1} f(z)g'(z)dz = [f(z)g(z)]\Big|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz \qquad (12.13)$$

12.7-Teorema. $z=\varphi(w)$ analitik funksiya S chiziqni o‘zaro bir qiymatli ravishda w tekislikdagi Γ chiziqqa akslantirsin, u holda

$$\int_C f(z)dz=\int_\Gamma f(\varphi(w))\varphi'(w)dw \qquad (12.14)$$

Eslatma. 1. Agar integrallash yo‘li aylana yoki uning biror yoyidan iborat bo‘lsa $z-z_0=\rho e^{i\theta}$ kabi almashtirish bajarish maqsadga muvofiq.

2. Agar integrallash chizig‘i yopiq konturdan iborat bo‘lsa, haqiqiy o‘zgaruvchili funksiyalar integrallaridagi kabi $\oint$ belgilashdan foydalanamiz.

12.11-Misol. a) Quyidagi integralni hisoblang:

$$I=\int_1^{1+i} zdz$$

►Integral ostidagi funksiya analitik bo‘lganligi uchun (12.12) formulaga ko‘ra

$$I=\int_1^{1+i} zdz=\left.\frac{z^2}{2}\right|_1^{1+i}=\frac{1}{2}((1+i)^2-1)=\frac{2i-1}{2}$$

b) Agar C integrallash chizig‘i $|z|=1, 0\le \arg z\le\pi$ aylana yoyidan iborat bo‘lsa, quyidagi integralni hisoblang

$$\int_C (z^2+z\bar{z})dz$$

Chiziq birlik aylanadan iborat bo‘lganligi uchun $z=e^{i\theta},\ dz=ie^{i\theta}d\theta$ kabi almashtirish bajaramiz, u holda

$$\int_0^\pi \left(e^{2i\theta}+1\right)ie^{i\theta}d\theta=i\int_0^\pi \left(e^{3i\theta}+e^{i\theta}\right)d\theta=\left.\left(\frac{1}{3}e^{3i\theta}+e^{i\theta}\right)\right|_0^\pi=$$
$$\frac{1}{3}e^{3i\pi}+e^{i\pi}-\frac{1}{3}-1=-\frac{8}{3}$$

c) $\int_C \frac{dz}{\sqrt{z}}$ integral hisoblansin. Bu yerda C chiziq $|z|=1$ aylananing yuqori yarim yoyi. Integral ostida ildizning $\sqrt{1}=-1$ shartni qanoatlantiruvchi tarmog‘i olinsin.

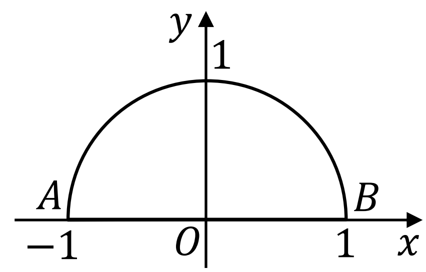
Bu integralda ham $z=e^{i\theta},\ dz=ie^{i\theta}d\theta$ kabi almashtirish bajaramiz va $\sqrt{1}=-1$ shartdan $\sqrt{e^{i\theta}}=e^{i\left(\frac{\theta}{2}+\pi\right)}$ ni hosil qilamiz. U holda

$$\int_C \frac{dz}{\sqrt{z}}=\int_0^\pi \frac{ie^{i\theta}d\theta}{e^{i\left(\frac{\theta}{2}+\pi\right)}}=i\int_0^\pi e^{i\left(\frac{\theta}{2}-\pi\right)}d\theta=\left.2e^{i\left(\frac{\theta}{2}-\pi\right)}\right|_0^\pi=2(1-i)$$

qiymatni hosil qilamiz. ◄

12.12-Misol. $\oint_C |z|\bar{z}dz$ integralni hisoblang, bu yerda C $-$yopiq kontur bo‘lib, $|z| = 1$ aylananing yuqori yarim qismi va $-1 \leq x \leq 1, y = 0$ kesmadan iborat. Integrallash yo‘nalishi soat strelkasi harakatiga qarama-qarshi yo‘nalishda.

►Berilgan integralni ikkita integralning yig‘indisi ko‘rinishida tasvirlaymiz. Birinchisini $|z| = 1$ aylananing yuqori yarim qismi bo‘lgan $C^{(1)}$ chiziq bo‘yicha, ikkinchisi esa Ox o‘qning $A(-1{,}0)$ va $B(1{,}0)$ nuqtalarini tutashtiruvchi $C^{(2)}$ kesma bo‘yicha olamiz (12.9-rasm):



12.9-rasm

$$\oint_C |z|\bar{z}dz = \int_{C^{(1)}} |z|\bar{z}dz + \int_{C^{(2)}} |z|\bar{z}dz.$$

Ularning harbirini alohida hisoblaymiz. Birinchi integraldagi $C^{(1)}$ yarim aylana $z = e^{it}, 0 \leq t \leq \pi$ parametrik tenglamaga ega. U holda $\bar{z} = e^{-it}$, $|z| = 1$ va $dz = ie^{it}dt$ bo‘ladi. Shuning uchun

$$\int_{C^{(1)}} |z|\bar{z}dz = \int_0^\pi 1 \cdot e^{-it} ie^{it} dt = i\int_0^\pi dt = \pi i.$$

$C^{(2)}$ kesmada $\bar{z} = x$ va $|z| = |x| = \begin{cases} -x, & \text{agar} -1 \leq x \leq 0, \\ x, & \text{agar } \ 0 \leq x \leq 1 \end{cases}$ ekanligini inobatga olsak

$$\int_{C^{(2)}} |z|\bar{z}dz = \int_{-1}^{0} (-x)xdx + \int_0^1 xxdx = -\frac{x^3}{3}\Bigg|_{-1}^{0} + \frac{x^3}{3}\Bigg|_0^1 = -\frac{1}{3} + \frac{1}{3} = 0$$

qiymatga ega bo‘lamiz.

Hisoblangan integrallarning qiymatlarini yuqoridagi tenglikga qo‘yib

$$\oint_C |z|\bar{z}dz = i\pi$$

berilgan integralning qiymatini hosil qilamiz.◄

Koshi teoremasi.

12.8-Teorema (Koshi). Agar $f(z)$ funksiya C yopiq kontur bilan chegaralangan bir bog‘lamli D soha va uning chegarasida analitik bo‘lsa, u holda

$$\oint_C f(z)dz = 0 \qquad (12.15)$$

► $f'(z)$ funksiya D sohada uzluksiz bo‘lsin. $f(z)$ funksiya analitik bo‘lganligi uchun uning haqiqiy va mavhum qismlari uchun Koshi-Riman shartlari bajariladi

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

(12.10) formulaga ko‘ra

$$\oint_C f(z)dz=\oint_C u(x,y)dx-v(x,y)dy+i\oint_C v(x,y)dx+u(x,y)dy$$

tenglik o‘rinli. Egri chiziqli integrallar nazariyasidan ma’lumki, Koshi-Rimanning ikkinchi shartiga ko‘ra

$$\oint_C u(x,y)dx-v(x,y)dy=0$$

birinchi shartiga ko‘ra

$$\oint_C v(x,y)dx+u(x,y)dy=0$$

Demak

$$\oint_C f(z)dz=0. \blacktriangleleft$$

Eslatma. Koshi teoremasi ko‘p bog‘lamli sohalar uchun ham o‘rinli. Bunday holda C bir necha yopiq konturlar birlashmasidan iborat bo‘ladi.

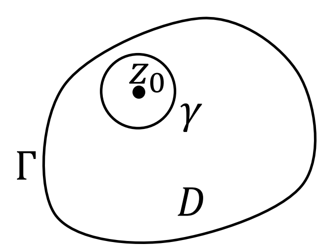
Koshi integrali. Koshining integral formulasi. Yuqorida isbot etilgan Koshi teoremasidan kompleks o‘zgaruvchili funksiyalar nazariyasining eng muhim formulalaridan biri- Koshining integral formulasi kelib chiqadi.

12.9-Teorema. $f(z)$ funksiya bir bog‘lamli D soha va uning musbat orientirlangan chegarasi C chiziqda analitik bo‘lsin. U holda D sohaning ichida yotuvchi ixtiyoriy z_0 nuqta uchun

$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \qquad (12.16)$$

formula o‘rinli.

► z_0 nuqtani o‘z ichiga oluvchi radiusi ρ ga teng bo‘lgan shunday γ aylana olamizki $|z-z_0|\le\rho$ doira to‘laligicha D sohada yotsin. Koshi teoremasiga ko‘ra (12.10- rasm)



12.10-rasm

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz+\frac{1}{2\pi i}\oint_{\gamma^-} \frac{f(z)}{z-z_0}dz=0$$

Bu yerda γ^- chiziqning manfiy orientirini, ya’ni soat strelkasi bo‘ylab yo‘nalishni anglatadi. Integrallarning 3- xossasiga ko‘ra

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}dz \qquad (12.17)$$

Demak (16) formulani γ kontur uchun isbotlash etarli.

$$\oint_\gamma \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = \oint_\gamma \frac{f(z)}{z-z_0}dz - f(z_0)\oint_\gamma \frac{1}{z-z_0}dz =$$

$$= \oint_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz \qquad (12.18)$$

Oxirgi integralda $f(z)$ uzluksiz bo‘lganligidan $\lim\limits_{\rho\to 0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = 0$ yoki

$$\lim_{\rho\to 0}\oint_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz = 0$$

Shuning uchun (12.18) tenglikda $\rho \to 0$ limitga o‘tsak, tenglikning chap tomonidagi integral (12.17) ga asosan ρ ga bog‘liq bo‘lmaganligi uchun

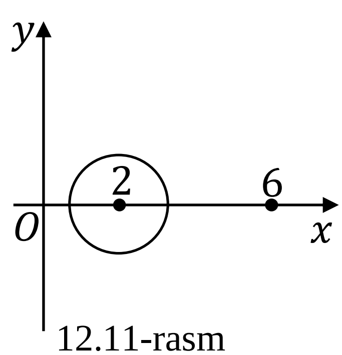
$$\oint_\gamma \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0$$

yoki (12.16) tenglikni hosil qilamiz. ◄

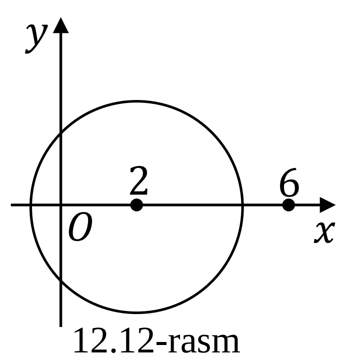
(12.16) tenglikning o‘ng tomonida turgan integralga *Koshi integrali*, tenglikning o‘ziga esa *Koshining integral formulasi* deb ataladi. Koshi integralini hisoblash uchun $f(z)$ funksiyaning soha chegarasidagi qiymatini bilish kerak.

O‘z navbatida Koshining integral formulasi, agar funksiyaning soha chegarasidagi qiymatlari ma’lum bo‘lsa, uning ixtiyoriy ichki nuqtasidagi qiymatini hisoblash imkonini beradi.

Eslatma. Koshining integral formulasi ko‘p bog‘lamli sohalar uchun ham o‘rinli bo‘lib, bu holda integral bir nechta integrallarga ajralib ularning barchasida chiziq bo‘ylab yurilganida soha chap tomonda qoladi.



12.11-rasm



12.12-rasm

12.12-Misol. Koshining integral formulasidan foydalanib integralni hisoblang

$$\oint_{C} \frac{e^{z^2}}{z^2-6z} dz$$

Agar

1) $C: |z-2| = 1$
2) $C: |z-2| = 3$
3) $C: |z-2| = 5$

bo‘lsa.

► 1)$|z-2| \leq 1$ yopiq doirani qaraymiz.

Bu sohada integral ostidagi funksiya analitik (12.11- rasm), shuning uchun Koshi teoremasiga ko‘ra,

$$\oint_{|z-2|=1} \frac{e^{z^2}}{z^2-6z} dz = 0$$

2) $|z-2| = 3$ aylananing ichida, maxraj nolga aylanadigan bitta nuqta bor (12.13- rasm). Bu $z = 0$ nuqta, shuning uchun Koshining integral formulasiga ko‘ra

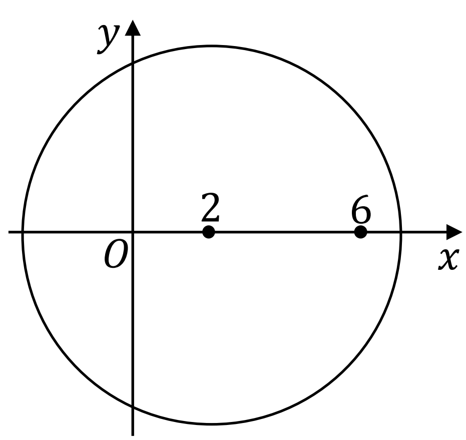
$$\oint_{|z-2|=3} \frac{e^{z^2}}{z^2-6z} dz = \oint_{|z-2|=3} \frac{\frac{e^{z^2}}{z-6}}{z} = 2\pi i f(0) = 2\pi i \frac{1}{0-6} = -\frac{\pi i}{3}$$

3) $|z-2| = 5$ chiziqning ichida maxraj nolga aylanadigan ikkala nuqta $z = 0$ va $z = 6$ lar ham yotadi (12.12- rasm), shuning uchun

$$\oint_{C} \frac{e^{z^2}}{z^2-6z} dz = \frac{1}{6}\oint_{C} \frac{e^{z^2}}{z-6} dz - \oint_{C} \frac{e^{z^2}}{z} dz = 2\pi i \frac{e^{36}}{6} - 2\pi i \frac{1}{6} =$$

$$= \frac{1}{3}\pi i(e^{36}-1)$$

bo‘ladi. ◄



12.13-rasm

12.10-Teorema. $f(z)$ funksiya bir bog‘lamli D soha va uning musbat orientirlangan chegarasi Γ chiziqda analitik bo‘lsin. U holda ixtiyoriy natural n soni uchun quyidagi formula o‘rinli

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)dz}{(z-z_0)^{n+1}} \qquad (12.19)$$

bu yerda $z_0 \in D, z \in \Gamma$.

Bu formulaga *yuqori tartibli hosilalar* formulasi deb ataladi. Uni hosil qilish uchun (12.16) Koshining integral formulasini z_0 ni o‘zgaruvchi deb qarab n marta differensiallash etarli. Bu yerda z_0 nuqta ixtiyoriy tanlanganligi uchun, analitik funksiya ixtiyoriy tartibdagi hosilaga ega degan xulosa kelib chiqadi. Yuqori tartibli hosilalar formulasi yordamida ba’zi integrallarni hisoblash mumkin.

12.13-Misol. Quyidagi integralni hisoblang.

$$\oint_{|z|=1} \frac{\cos z}{z^3} dz$$

►Bu misolda $f(z) = \cos z$, $z = 0$ nuqta esa qaralayotgan chiziqning ichida yotadi, shuning uchun (12.19) formulaga asosan integral

$$\oint_{|z|=1} \frac{\cos z}{z^3} dz = \frac{2\pi i}{2!}(\cos z)''\Big|_{z=0} = \frac{2\pi i}{2}(-1) = -\pi i$$

qiymatga teng bo‘ladi. ◄

12.14-Misol. Quyidagi integralni hisoblang.

$$\oint_{|z-3|=6} \frac{z}{(z-2)^2(z+4)} dz$$

►$z = 2$ nuqta integrallash chizig‘i $|z-3| = 6$ aylananing ichida, $z = -4$ esa tashqarisida yotadi. Shuning uchun integral ostidagi funksiyani shakl almashtirib (12.19) formulani qo‘llaymiz va natjada:

$$\oint_{|z-3|=6} \frac{z}{(z-2)^2(z+4)} dz = \oint_{|z-3|=6} \frac{\frac{z}{z+4}}{(z-2)^2} dz =$$

$$= \frac{2\pi i}{1!}\left(\frac{z}{z+4}\right)'\Big|_{z=2} = 2\pi i \frac{4}{(z+4)^2}\Big|_{z=2} = \frac{2\pi i}{9}$$

qiymatni hosil qilamiz.◄

12.5. Qatorlar va maxsus nuqtalar

Sonli qatorlar. Hadlari kompleks sonlardan iborat bo‘lgan qatorni kompleks sonlar qatori deb yuritiladi.

Agar quyidagicha kompleks sonlar qatori qaralayotgan bo‘lsa

$$\sum_{n=1}^{\infty}(\alpha_n+i\beta_n)=$$
$$=(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)+\cdots+(\alpha_n+i\beta_n)+\cdots \qquad (12.20)$$

haqiqiy sonlar qatorida bo‘lgani kabi (12.20) qatorning dastlabki n ta hadining yig‘indisi uning xususiy yig‘indisi deb atalib,

$$S_n=\sum_{k=1}^{n}(\alpha_k+i\beta_k)$$

bilan belgilaymiz.

Ta’rifga ko‘ra n cheksizga intilganda hususiy yig‘indilar biror chekli limitga ega bo‘lsa, uni (12.20) qatorning yig‘indisi deb atalib, (12.20) qatorning o‘zini yaqinlashuvchi, aksincha esa uzoqlashuvchi deb ataladi.

12.11-Teorema. (12.20) kompleks sonlar qatori yaqinlashuvchi bo‘lishi uchun, uning hadlari haqiqiy va mavhum qismlaridan tuzilgan

$$\sum_{n=1}^{\infty}\alpha_n \text{ va } \sum_{n=1}^{\infty}\beta_n$$

haqiqiy sonlar qatorlari bir vaqtda yaqinlashuvchi bo‘lishi zarur va etarli.

► **Zarurligi.** Faraz qilaylik (12.20) qator yaqinlashuvchi bo‘lsin. U holda ta’rifga ko‘ra

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sum_{k=1}^{n}(\alpha_k+i\beta_k)=\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_k+i\lim_{n\to\infty}\sum_{k=1}^{n}\beta_k$$

chekli son bo‘lishi kerak. Bu yerdan tenglikning o‘ng tomonidagi ikkala limit ham chekli bo‘lishi zarurligi kelib chiqadi.

Yetarliligi. (12.20) qatordan tuzilgan quyidagi

$$\sum_{n=1}^{\infty}\alpha_n=\alpha_1+\alpha_2+\cdots+\alpha_n+\cdots$$
$$\sum_{n=1}^{\infty}\beta_n=\beta_1+\beta_2+\cdots+\beta_n+\cdots$$

haqiqiy sonlar qatorlarining har biri alohida-alohida yaqinlashuvchi qatorlar bo‘lsin. U holda agar ularning xususiy yig‘indilarini mos ravishda σ_n va τ_n orqali belgilasak, quyidagilarni yozish mumkin bo‘ladi:

$$\lim_{n\to\infty}\sigma_n=\sigma \text{ (chekli) va } \lim_{n\to\infty}\tau_n=\tau \text{ (chekli)}$$

Shuningdek agar $S_n = \sigma_n + i\tau_n$ ekanligini nazarda tutsak

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\sigma_n + i\tau_n) = \sigma + i\tau \ \text{(chekli)}$$

Demak (20) qator yaqinlashuvchi ekan. ◄

12.21-Ta'rif Agar (12.20) kompleks sonlar qatoriga mos keluvchi quyidagi

$$\sum_{n=1}^{\infty}|\alpha_n + i\beta_n| = |\alpha_1 + i\beta_1| + |\alpha_2 + i\beta_2| + \cdots + |\alpha_n + i\beta_n| + \cdots$$

qator yaqinlashuvchi bo'lsa, (12.20) qatorning o'zini mutlaq yaqinlashuvchi qator deb yuritiladi.

Eslatma. Kompleks hadli qatorlarga haqiqiy hadli qatorlar uchun qo'llanilgan yaqinlashish belgilarini qo'llash mumkin bo'lib, bu holda kompleks hadli qatorlar mutlaq yaqinlashishga tekshiriladi.

12.15-Misol. Quyidagi qatorni yaqinlashishga tekshiring

$$\sum_{n=1}^{\infty}\frac{\cos in}{2^n}$$

►Trigonometrik funksiyalarning aniqlanishidan $\cos in = (e^n + e^{-n})/2$ bo'ladi. Qator yaqinlashishining Dalamber belgisiga ko'ra

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}+e^{-n-1}}{2\cdot 2^{n+1}}\cdot\frac{2\cdot 2^n}{e^n+e^{-n}} =$$

$$= \lim_{n\to\infty}\frac{(e+e^{-2n-1})}{2(1+e^{-2n})} = \frac{e}{2} > 1$$

Demak qator uzoqlashuvchi. ◄

Darajali qatorlar. Hadlari kompleks o'zgaruvchili funksiyalardan iborat

$$f_0(z) + f_1(z) + \cdots + f_n(z) + \cdots \tag{12.21}$$

qatorga *funktsional qator* deyiladi.

Bu ko'rinishdagi funktsional qatorlar haqiqiy o'zgaruvchili funksiyalar qatorlari bilan bir xossalarga ega bo'lganligi uchun biz ularni qoldiramiz (XI bob). Funktsional qatorlarning muhim xususiy hollaridan biri darajali qatorlar bo'lib, bunday qatorlar kompleks tekisligida qaraladi.

$$\sum_{n=1}^{\infty}c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \tag{12.22}$$

ko'rinishdagi qatorga *darajali qator* deyiladi, bu yerda $c_n -$ o'zgarmas kompleks son. Bu qator z ning ba'zi qiymatlarida yaqinlashuvchi, ba'zilarida esa uzoqlashuvchi bo'ladi. Demak, kompleks tekisligi ikki qismga ajralib, uning bir qismida qator yaqinlashadi, ikkinchisida uzoqlashadi. Darajali qatorlarning yaqinlashish sohasini aniqlaydigan quyidagi teoremani keltiramiz.

12.12-Teorema. (Abel) Agar (12.22) darajali qator biror $z_0 \neq 0$ nuqtada yaqinlashsa, u holda bu qator $|z| < |z_0|$ tengsizlikni qanoatlantiruvchi barcha z nuqtalarda yaqinlashuvchi bo'ladi va aksincha (12.22) qator biror z_0 da uzoqlashuvchi bo'lsa, $|z| > |z_0|$ tengsizlikni qanoatlantiruvchi barcha z nuqtalarda uzoqlashuvchi bo'ladi.

Abel teoremasidan (12.22) darajali qatorning yaqinlashish sohasi doiradan iborat ekanligi kelib chiqadi, chunki tekislikda $|z| < |z_0|$ soha markazi nolda radiusi $|z_0|$ ga teng bo'lgan doirani ifodalaydi. $|z_0|$ sonlarining eng kattasiga darajali qatorning *yaqinlashish radiusi* deb atalib, uni

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}} \qquad (12.23)$$

formulalar yordamida hisoblanadi. $|z| < R$ sohaga darajali qatorning *yaqinlashish doirasi* deb ataladi.

$|z_0| = R$ aylana ustidagi nuqtalarda qator yaqinlashishi ham uzoqlashishi ham mumkin. $R = 0$ da qator faqatgina $z = 0$ nuqtada yaqinlashadi, $R = \infty$ da esa kompleks tekisligining barcha nuqtalarida yaqinlashadi.

12.16-Misol. Quyidagi darajali qatorning yaqinlashish sohasini toping.

$$\sum_{n=1}^{\infty}(1+i)^n z^n$$

► $1 + i = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$ bo'lganligi uchun

$$c_n = (1+i)^n = \sqrt{2^n}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right), \quad |c_n| = \sqrt{2^n}$$

Demak (23) formulaga ko'ra

$$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{\sqrt{2^n}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Shunday qilib qatorning yaqinlashish sohasi markazi nolda radiusi $R = \sqrt{2}/2$ bo'lgan doiradan iborat. ◄

Abel teoremasidan quyidagi bir necha muhim natijalar kelib chiqadi.

1-Natija. Yaqinlashish doirasining ichida darajali qator analitik funksiyaga yaqinlashadi.

2-Natija. Darajali qatorni yaqinlashish doirasi ichida ixtiyoriy marta hadlab differensiallash va integrallash mumkin bo‘lib, hosil bo‘lgan qatorlarning yaqinlashish radiusi berilgan qator yaqinlashish radiusiga teng.

Eslatma. (12.22) darajali qator bilan birgalikda

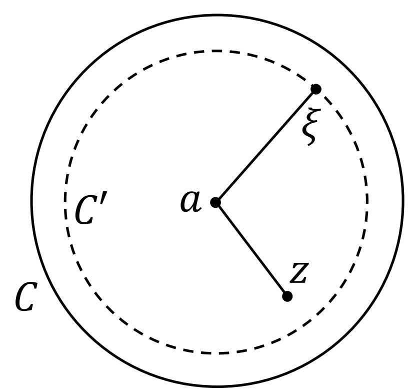
$$\sum_{n=1}^{\infty} s_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots + c_n(z-a)^n + \cdots$$

ko‘rinishdagi qatorlar ham darajali qatorlardir. Ravshanki, u ham doirada yaqinlashuvchi bo‘lib, yaqinlashish radiusi (12.23) formulalar bilan topiladi. Yaqinlashish sohasi $|z-a| < R$ doiradan iborat.

Teylor qatori. Markazi $z = a$ nuqtada bo‘lgan C aylana bilan chegaralangan D doira ichida bir qiymatli analitik $f(z)$ funksiyani qaraymiz. Bu funksiyani $z = a$ nuqta atrofida darajali qatorga yoyamiz. z nuqta D doira ichida ixtiyoriy joylashgan bo‘lsin (12.14 rasm).

► D doiraning ichida markazi a nuqtada bo‘lgan shunday C' aylana olamizki, z nuqta uning ichida yotsin. Agar ξ nuqta C' aylananing ustidagi ixtiyoriy nuqta bo‘lsa, u holda Koshining integral formulasiga ko‘ra

$$f(z) = \frac{1}{2\pi i}\int_{C'} \frac{f(\xi)d\xi}{\xi - z} \qquad (12.24)$$



12.14 - rasm

$\frac{1}{\xi - z}$ kasrni quyidagicha shakl almashtiramiz

$$\frac{1}{\xi - z} = \frac{1}{\xi - a - (z-a)} = \frac{1}{(\xi - a)\left(1 - \frac{z-a}{\xi - a}\right)}$$

$|\xi - a|$ modul C' aylananing radiusiga teng, $|z-a|$ modul esa z nuqtadan a nuqtagacha bo‘lgan masofaga teng bo‘lib, $|\xi - a|$ dan kichik, shuning uchun $\left|\frac{z-a}{\xi-a}\right| < 1$. O‘z navbatida $\frac{1}{1-\frac{z-a}{\xi-a}}$ ifodani C' aylananing ichida joylashgan ixtiyoriy z nuqtada yaqinlashuvchi cheksiz kamayuvchi geometrik progressiyaning yig‘indisi deb qarash mumkin:

$$\frac{1}{1-\frac{z-a}{\xi-a}}=1+\frac{z-a}{\xi-a}+\left(\frac{z-a}{\xi-a}\right)^2+\cdots+\left(\frac{z-a}{\xi-a}\right)^n+\cdots \qquad (12.25)$$

O‘zgarmas z uchun $\left|\frac{z-a}{\xi-a}\right|=q$ kattalik C' aylanada o‘zgarmas son bo‘lganli uchun (12.25) qator bu aylanada ξ ga nisbatan yaqinlashuvchi bo‘ladi. Shuning uchun (12.25) qator hadlarining modullari

$$1+q+q^2+\cdots+q^n+\cdots$$

qator hadlari bilan ustma-ust tushadi.

Demak, oxirgi qator (12.25) qator uchun majoranta bo‘ladi. O‘z navbatida darajali qatorlarning xossalariga ko‘ra (12.25) qatorni biror songa hadlab ko‘paytirish, uni hadlab integrallash mumkin bo‘ladi. Yuqoridagilarga asosan (12.25) tenglikning ikkala tomonini $\frac{f(\xi)}{\xi-a}$ nisbatga hadlab ko‘paytiramiz, natijada

$$\frac{f(\xi)}{\xi-z}=\frac{f(\xi)}{\xi-a}+\frac{(z-a)f(\xi)}{(\xi-a)^2}+\cdots+\frac{(z-a)^nf(\xi)}{(\xi-a)^{n+1}}+\cdots$$

qatorga ega bo‘lamiz. Agar bu qatorni $\frac{1}{2\pi i}$ ga ko‘paytirib, C' chiziq bo‘yicha hadlab integrallasak, $f(z)$ funksiyaning $z-a$ ning darajalari bo‘yicha qatorga yoyilmasini hosil qilamiz

$$f(z)=\frac{1}{2\pi i}\int_{C'}\frac{f(\xi)d\xi}{\xi-a}+\frac{z-a}{2\pi i}\int_{C'}\frac{f(\xi)d\xi}{(\xi-a)^2}+\cdots+\frac{(z-a)^n}{2\pi i}\int_{C'}\frac{f(\xi)d\xi}{(\xi-a)^{n+1}}$$
$$+\cdots=$$
$$=c_0+c_1(z-a)+c_2(z-a)^2+\cdots c_n(z-a)^n+\cdots$$

bu yerda

$$c_n=\frac{1}{2\pi i}\int_{C'}\frac{f(\xi)d\xi}{(\xi-a)^{n+1}} \quad (n=0,1,2\ldots) \qquad (12.26)$$

Shunday qilib quyidagi teoremani isbot qildik. ◄

12.13-Teorema. Agar $f(z)$ markazi $z=a$ nuqtada bo‘lgan D doira ichida bir qiymatli analitik funksiya bo‘lsa, u holda bu funksiyani D doirada yaqinlashuvchi bo‘lgan darajali qatorga yoyish mumkin bo‘lib, uning koeffitsientlari (12.26) formulalar bilan hisoblanadi.

Yuqorida hosil qilingan qatorda, uning (12.26) koeffitsientlarini hisoblashda C' chiziq sifatida, D doiraning ichida yotuvchi markazi $z=a$

nuqtada bo‘lgan ixtiyoriy aylana yoki a nuqtani o‘z ichiga olib uni bir marta aylanuvchi ixtiyoriy yopiq konturni olishimiz mumkin. Chunki Koshining integral formulasiga asosan integralning qiymati kontur tanlanishiga bog‘liq emas.

Agar (12.19) yuqori tartibli hosilalar formulasini e’tiborga olsak

$$c_0 = \frac{1}{2\pi i}\int_{C'} \frac{f(\xi)d\xi}{\xi - a} = f(a),$$

$$c_n = \frac{1}{2\pi i}\int_{C'} \frac{f(\xi)d\xi}{(\xi - a)^{n+1}} = \frac{f^{(n)}(a)}{n!} \quad (n = 0,1,2 \dots)$$

tengliklarga ega bo‘lamiz. Ular yordamida matematik analizdan ma’lum bo‘lgan Teylor formulasini hosil qilamiz

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \\ + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{12.27}$$

$a = 0$ bo‘lsa, ma’lum bo‘lgan Makloren qatorini hosil qilamiz

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots \tag{12.28}$$

Makloren formulalardan foydalangan holda ba’zi elementar funksiyalarning darajali qatorlarga yoyilmasini keltirib chiqarish mumkin. Buning uchun funksiyani bevosita (12.28) formulalarga qo‘yish etarli:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots + \frac{z^n}{n!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \cdots \\ + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}z^n + \cdots$$

Bu qatorlarning birinchi uchtasi kompleks tekisligining barcha nuqtalarida yaqinlashuvchi bo‘ladi, oxirgisi esa binomial qator deb atalib $|z| < 1$ doirada yaqinlashuvchi bo‘ladi.

Xususiy $\alpha = -1$ bo‘lgan holda binomial qator

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots$$
ko‘rinishni oladi.

12.17-Misol. $f(z) = 1/(1-3z)$ funksiyani $z-1$ ayirmaning darajalari bo‘yicha qatorga yoying.

►Berilgan funksiyaning ko‘rinishini o‘zgartiramiz:
$$f(z) = \frac{1}{1-3z} = \frac{1}{-2-3(z-1)} = -\frac{1}{2}\frac{1}{1+\frac{3}{2}(z-1)}.$$
Yuqorida keltirilgan $1/(1+z)$ funksiyaning yoyilmasida z o‘rniga $\frac{3}{2}(z-1)$ ifodani qo‘ysak
$$\frac{1}{5-3z} = -\frac{1}{2}\left(1 - \frac{3}{2}(z-1) + \frac{3^2}{2^2}(z-1)^2 - \frac{3^3}{2^3}(z-1)^3 + \cdots\right) =$$
$$-\frac{1}{2} + \frac{3}{2^2}(z-1) - \frac{3^2}{2^3}(z-1)^2 + \frac{3^3}{2^4}(z-1)^3 + \cdots$$
yoyilmani hosil qilamiz. Bu qator
$$\left|\frac{3}{2}(z-1)\right| < 1 \quad \text{yoki} \quad |z-1| < \frac{2}{3}$$
bo‘lganda yaqinlashadi, ya’ni qatorning yaqinlashish radiusi $R = 2/3$ bo‘ladi.◄

Loran qatori.
$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (12.29)$$
ko‘rinishdagi qatorga Loran qatori deb ataladi, bu yerda a, c_n - kompleks sonlar, z -kompleks o‘zgaruvchi.
$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} \qquad (12.30)$$
qatorning *bosh qismi*,
$$\sum_{n=0}^{\infty} c_n (z-a)^n \qquad (12.31)$$
esa *to‘g‘ri qismi* deyiladi.

Ravshanki, (12.29) qator yaqinlashuvchi bo‘lishi uchun (12.30) va (12.31) qatorlar bir vaqtda yaqinlashuvchi bo‘lishi kerak. (12.31) darajali qatorning yaqinlashish sohasi doira bo‘lib, uning radiusi (12.23) formula bilan hisoblanadi. (12.30) qatorning yaqinlashish sohasini topish uchun bu qatorda

$$\frac{1}{(z-a)^n} = \xi^n$$

almashtirish bajaramiz va

$$\sum_{n=1}^{\infty} c_{-n}\xi^n$$

ni hosil qilamiz. U bizga ma'lum darajali qator bo‘lib uning yaqinlashish radiusini

$$\frac{1}{r} = \lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-n-1}}\right| = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|c|}}$$

formula bilan hisoblanadi. Demak, agar

$$\frac{1}{|z-a|} < \frac{1}{r} \text{ yoki } \quad |z-a| > r$$

bo‘lsa, (12.30) qator yaqinlashuvchi bo‘ladi.

Yuqorida ta'kidlaganimizdek, Loran qatori yaqinlashuvchi bo‘lishi uchun uning to‘g‘ri va bosh qismlari bir vaqda yaqinlashuvchi bo‘lishi kerak. $r < R$ bo‘lsa (12.30) va (12.31) qatorlarning umumiy yaqinlashish sohasi mavjud va Loran qatori

$$r < |z-a| < R$$

xalqada yaqinlashuvchi bo‘ladi.

Agar $r > R$ bo‘lsa Loran qatori bironta ham nuqtada yaqinlashmaydi. $r = R$ bo‘lganida Loran qatori aylananing ba'zi nuqtalarida yaqinlashuvchi, qolgan nuqtalarida uzoqlashuvchi bo‘lishi mumkin.

12.18-Misol. Quyidagi qatorning yaqinlashish sohasi aniqlansin.

$$\sum_{n=1}^{\infty}\frac{(5+12i)^n}{(z-5i)^n} + \sum_{n=0}^{\infty}\frac{(z-5i)^n}{17^n}$$

►Har bir qatorning yaqinlashish sohalarini alohida aniqlaymiz:

$$r = \lim_{n\to\infty}\sqrt[n]{|(5+12i)^n|} = \lim_{n\to\infty}|5+12i| = 13.$$

Demak, Loran qatorining bosh qismi $|z-5i| > 13$ da yaqinlashadi.

$$R = \lim_{n\to\infty}\sqrt[n]{\left|1/\left(\frac{1}{17^n}\right)\right|} = \lim_{n\to\infty}|17| = 17$$

Loran qatorining to‘g‘ri qismi $|z-5i| < 17$ doirada yaqinlashar ekan. Shunday qilib berilgan qatorning yaqinlashish sohasi $13 < |z-5i| < 17$ xalqadan iborat bo‘lar ekan. ◄

12.14-Teorema. Agar $f(z)$ funksiya $r < |z-a| < R$ xalqada analitik bo‘lsa, u holda $f(z)$ funksiyani bu xalqada yagona usulda Loran qatoriga yoyish mumkin va uning koeffitsientlari

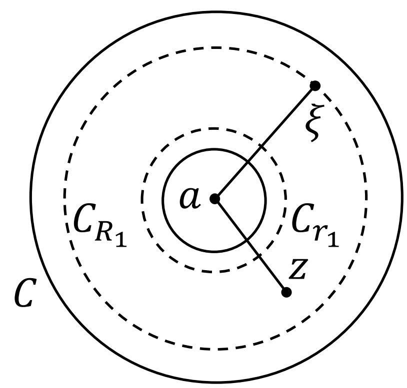
$$c_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-a)^{n+1}}d\xi, \qquad n = 0, \pm 1, \pm 2, \dots$$

formulalardan topiladi, bu yerda C markazi a nuqtada bo‘lgan berilgan xalqaning ichida yotuvchi ixtiyoriy aylana.

► z nuqtani $r < |z-a| < R$ xalqada ixtiyoriy tanlab mahkamlab olamiz. Bu xalqada radiuslari $r < r_1 < R_1 < R$ tengsizliklarni qanoatlantiruvchi ikkita C_{r_1} va C_{R_1} aylanalar olamiz. Ko‘p bog‘lamli sohalar uchun Koshi formulasiga ko‘ra

$$f(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\xi)d\xi}{\xi - z} + \frac{1}{2\pi i}\int_{C_{r_1}^-} \frac{f(\xi)d\xi}{\xi - z}, \tag{12.32}$$

bu yerda birinchi integralda yo‘nalish soat strelkasiga qarama-qarshi, ikkinchisida soat strelkasi bo‘ylab (12.15 rasm).



12.15 - rasm

C_{R_1} aylanada $\left|\frac{z-a}{\xi-a}\right| < 1$ bo‘lganligi uchun Teylor formulasi keltirib chiqarilishida bajarilgan amallarni takrorlasak

$$\frac{f(\xi)}{\xi - z} = \frac{f(\xi)}{\xi - a} + \frac{(z-a)f(\xi)}{(\xi-a)^2} + \dots + \frac{(z-a)^n f(\xi)}{(\xi-a)^{n+1}} + \cdots$$

tenglikni hosil qilamiz. Uni C_{R_1} aylana bo‘ylab hadlab integrallasak

$$\frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\xi)d\xi}{\xi - z} = \sum_{n=1}^{\infty} c_n (z-a)^n \tag{12.33}$$

kabi darajali qator hosil bo‘lib, uning koeffitsientlari quyidagicha topiladi

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-a)^{n+1}}d\xi, \qquad n = 0,1,2, \dots$$

C_{r_1} aylanada esa $\left|\frac{\xi-a}{z-a}\right| < 1$ bo'lganligi uchun

$$\frac{1}{\xi - z} = -\frac{1}{z - a - (\xi - a)} = -\frac{1}{(z-a)\left(1 - \frac{\xi - a}{z - a}\right)}$$

$$= -\frac{1}{(z-a)}\sum_{n=1}^{\infty}\left(\frac{\xi - a}{z - a}\right)^n$$

Uni $\frac{f(\xi)}{2\pi i}$ ifodaga ko'paytirib C_{r_1} aylanada hadlab integrallasak

$$\frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\xi)d\xi}{\xi - z} = -\sum_{n=1}^{\infty}\frac{c_{-n}}{(z-a)^n} \qquad (12.34)$$

bu yerda

$$c_{-n} = \frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\xi)}{(\xi - a)^{-n+1}}d\xi, \qquad n = 1,2,\dots$$

(12.33) va (12.34) qatorlar koeffitsientlarni hisoblash formulalarida Koshi teoremasiga asosan integrallash chiziqlarini $r < |z-a| < R$ xalqada yotuvchi va markazi a nuqtada bo'lgan ixtiyoriy aylana bilan almashtirish mumkin. Agar (12.33) va (12.34) tengliklarni (12.32) formulaga qo'ysak

$$f(z) = \sum_{n=0}^{\infty}c_n(z-a)^n + \sum_{n=1}^{\infty}\frac{c_{-n}}{(z-a)^n} = \sum_{n=-\infty}^{\infty}c_n(z-a)^n \qquad (12.35)$$

kabi Loran qatoriga yoyilmani hosil qilamiz. Bu yerda

$$c_n = \frac{1}{2\pi i}\int_C\frac{f(\xi)}{(\xi - a)^{n+1}}d\xi, \qquad n = 0, \pm 1, \pm 2, \dots$$

C chiziq $r < |z-a| < R$ xalqada yotuvchi va markazi a nuqtada bo'lgan aylana.

Bu yerda ham C aylanani $r < |z-a| < R$ xalqada yotuvchi va a nuqta ichida bo'lgan ixtiyoriy yopiq kontur bilan almashtirish mumkin. ◄

12.19-Misol. $f(z) = 1/(z+2)$ funksiyani $z = 0$ nuqtaning atrofida darajali qatorga yoying.

► Berilgan funksiyaning n tartibli hosilasi formulasini topamiz

$$f^{(n)}(z) = \left(\frac{1}{z+2}\right)^{(n)} = [(z+2)^{-1}]^{(n)} =$$

$$= (-1)\cdot(-2)\cdot\ldots\cdot(-n)\cdot(z+2)^{-n-1},$$

hamda bu hosilaning nol nuqtadagi qiymatini topamiz

$$f^{(n)}(0) = (-1)^n\cdot n!\,2^{-n-1}$$

U holda $c_n=\frac{f^{(n)}(0)}{n!}=(-1)^n\cdot 2^{-n-1}$ ekanligini nazarda tutsak, funksiyaning Teylor qatoriga yoyilmasi quyidagi ko‘rinishda bo‘ladi

$$f(z)=\frac{1}{z+2}=\sum_{n=0}^{\infty}(-1)^n\cdot 2^{-n-2}\cdot z^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$$

Bu qatorning yaqinlashish radiusini topamiz

$$R=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{C_n}}=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{(-1)^n 2^{(-n-1)}}}=\frac{1}{2^{-1}\lim\limits_{n\to\infty}\sqrt[n]{2^{-1}}}=2$$

Demak, qatorning yaqinlashish sohasi $|z|<2$ doiradan iborat ekan. ◄

12.20-Misol. $f(z)=\frac{1}{(z^2-1)^2}$ funksiyani $0<|z-1|<2$ xalqada Loran qatoriga yoying.

►Berilgan $f(z)$ funksiyani $(z-1)$ ning musbat va manfiy darajalari bo‘yicha yig‘indisi ko‘rinishida ifodalashimiz kerak. Buning uchun funksiyani quyidagicha shakl almashtiramiz

$$\frac{1}{(z^2-1)^2}=\frac{1}{4}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)^2=$$

$$=\frac{1}{4}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right).$$

Birinchi ikkita had kerakli ko‘rinishga ega. Qolgan hadlarni shakl almashtiramiz.

$|z-1|<2$ bo‘lganligi uchun

$$\frac{1}{z+1}=\frac{1}{z-1+2}=\frac{1}{2}\frac{1}{1+\frac{z-1}{2}}=$$

$$=\frac{1}{2}\left(1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3\ldots+(-1)^n\left(\frac{z-1}{2}\right)^n+\cdots\right)$$

To‘rtinchi hadni shakl almashtiramiz

$$\frac{1}{(z+1)^2}=\frac{1}{4}\left[1+\left(\frac{z-1}{2}\right)\right]^{-2}$$

bu yerda ham $\left|\frac{z-1}{2}\right|<1$ bo‘lganligi uchun, binomial yoyilmadan foydalanib qatorga yoyamiz

$$\frac{1}{(z+1)^2}=\frac{1}{4}\left[1-2\cdot\frac{z-1}{2}+\frac{-2(-2-1)}{2!}\left(\frac{z-1}{2}\right)^2+\right.$$

$$+\frac{-2(-2-1)(-2-2)}{3!}\left(\frac{z-1}{2}\right)^3+\cdots\Bigg]$$

Barcha yoyilmalarni $f(z)$ funksiyaga qo‘ysak

$$f(z)=\frac{1}{(z^2-1)^2}=\frac{1}{4}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right)=$$

$$=\frac{1}{4}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}\right)+\frac{3}{16}-\frac{1}{8}(z-1)+\frac{5}{64}(z-1)^2-$$

$$-\frac{3}{64}(z-1)^3+\cdots$$

ko‘rinishdagi Loran qatoriga yoyilmani hosil qilamiz. ◄

Yakkalangan maxsus nuqtalar.

12.22-Ta’rif. Agar

$$f(z_0)=0\,,\ f'(z_0)=0\,,\ldots,f^{(n-1)}(z_0)=0\,,f^{(n)}(z_0)\neq 0$$

munosabatlar o‘rinli bo‘lsa, z_0 nuqta $f(z)$ funksiyaning n – *tartibli noli* deb ataladi, va $n=1$ bo‘lsa, z_0 *oddiy nol* deyiladi.

Masalan, $z_0=0$ nuqta $f(z)=\sin z$ funksiya uchun oddiy nol, $\varphi(z)=z\sin z$ funksiya uchun esa ikkinchi tartibli nol bo‘ladi.

12.15-Teorema. z_0 nuqta $f(z)$ analitik funksiyaning m tartibli noli bo‘lishi uchun funksiyani z_0 nuqtaning atrofida

$$f(z)=(z-z_0)^m\varphi(z)$$

ko‘rinishda ifodalash mumkin bo‘lishi zarur va yetarli. Bu yerda $\varphi(z)$ funksiya z_0 nuqtada analitik va $\varphi(z_0)\neq 0$.

► $f(z)$ funksiya z_0 nuqtaning atrofida analitik bo‘lganligi uchun uni darajali qatorga yoyish mumkin

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$

$$+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\cdots$$

Agar z_0 nuqta m tartibli nol bo‘lsa, yuqoridagi yig‘indida dastlabki $m-1$ ta had nolga teng bo‘ladi. U holda

$$f(z)=\frac{f^{(m)}(z_0)}{m!}+\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1}+\cdots=$$

$$=(z-z_0)^m\left[\frac{f^{(m)}(z_0)}{m!}+\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)+\cdots\right]=(z-z_0)^m\varphi(z)$$

bu yerda, $\varphi(z)$ funksiya z_0 nuqtada analitik va $\varphi(z_0) \neq 0$. Teskari tasdiq

$$f(z) = (z - z_0)^m \varphi(z)$$

funksiyadan bevosita hosila olish bilan tekshiriladi. ◄

12.23-Ta'rif. Agar $z = z_0$ nuqtaning shunday atrofi mavjud bo'lib, bu atrofda funksiya $z = z_0$ nuqtadan tashqari barcha nuqtalarda analitik bo'lsa, z_0 nuqta $f(z)$ funksiyaning yakkalangan maxsus nuqtasi deyiladi.

$z = z_0$ yakkalangan maxsus nuqta bo'lsa, yuqorida isbot qilingan teoremaga ko'ra $f(z)$ funksiyani $0 < |z - z_0| < R$ xalqada

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

ko'rinishda Loran qatoriga yoyish mumkin. Bu yoyilmaning bosh qismi uchun uch xil holat bo'lishi mumkin:

a) Loran qatorida $z - z_0$ ning manfiy darajalari ishtirok etmaydi;

b) Yoyilmaning tarkibida $z - z_0$ ning chekli sondagi manfiy darajalari ishtirok etadi;

s) Yoyilmada $z - z_0$ ning cheksiz sondagi manfiy darajalari ishtirok etadi

Yuqoridagilarga ko'ra funksiyaning maxsus nuqtalari uch turga bo'linadi. Biz bu hollarning har birini alohida qarab chiqamiz.

1. Funksiyaning z_0 nuqta atrofidagi Loran qatoriga yoyilmasida manfiy darajali hadlar yo'q, ya'ni

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Bunday nuqtaga ***tuzatib bo'ladigan maxsus nuqta*** deb ataladi. Ko'rinib turibdiki, $z \to z_0$ da funksiyaning chekli limiti mavjud va $\lim_{z \to z_0} f(z) = c_0$. Agar funksiya z_0 nuqtada aniqlanmagan yoki aniqlansa ham qiymati boshqa son bo'lsa, unga z_0 nuqtada c_0 qiymat bersak, ya'ni $f(z_0) = c_0$ deb olsak funksiya analitik bo'lib qoladi. Bajarilgan amal yordamida funksiyaning uzilishini tuzatdik. Ana shu sababga ko'ra bunday maxsus nuqtani tuzatib bo'ladigan deyiladi.

Shunday qilib quyidagi teoremani isbotladik.

12.16-Teorema. Agar z_0 nuqta $f(z)$ funksiyaning tuzatib bo'ladigan maxsus nuqtasi bo'lsa, u holda $\lim_{z \to z_0} f(z) = c_0$ chekli limit mavjud.

2. Funksiyaning z_0 nuqta atrofidagi Loran qatoriga yoyilmasida chekli sondagi manfiy darajali hadlar ishtirok etadi, ya'ni

$$f(z)=\sum_{n=-m}^{\infty} c_n(z-z_0)^n$$

bo‘lsa z_0 funksiyaning ***qutb maxsus nuqtasi*** deyiladi. Agar bu yerda $c_{-m}\neq 0$ bo‘lsa, z_0 ga m tartibli qutb maxsus nuqta deb ataladi. $m=1$ da *oddiy qutb* deyiladi.

12.17-Teorema. Agar z_0 funksiyaning qutb maxsus nuqtasi bo‘lsa, u holda

$$\lim_{z\to z_0} f(z)=\infty$$

► $f(z)$ funksiyaning Loran qatorini qaraylik

$$\frac{c_{-m}}{(z-z_0)^m}+\frac{c_{-m+1}}{(z-z_0)^{m-1}}+\cdots+\frac{c_{-1}}{z-z_0}+\sum_{n=0}^{\infty} c_n(z-z_0)^n=$$

$$=(z-z_0)^{-m}[c_{-m}+c_{-m+1}(z-z_0)+\cdots+c_{-1}(z-z_0)^{m-1}]$$

$$+\sum_{n=0}^{\infty} c_n(z-z_0)^n=$$

$$=(z-z_0)^{-m}\psi(z)+\sum_{n=0}^{\infty} c_n(z-z_0)^n \qquad (12.36)$$

Ravshanki, $\psi(z)$ funksiya z_0 nuqtaning atrofida chegaralangan va analitik, shuning uchun $\lim\limits_{z\to z_0} f(z)=\infty$. (12.36) tenglikni shakl almashtirib funksiyani

$$f(z)=\frac{\varphi(z)}{(z-z_0)^m}$$

ko‘rinishda yozib olsak bo‘ladi, bu yerda $\varphi(z)$ $(\varphi(z_0)\neq 0)$ analitik funksiya. Funksiyaning bu ko‘rinishdagi ifodasi yordamida qutbning tartibini ham aniqlash mumkin. ◄

Shuni ta’kidlashimiz lozimki, funksiyaning nollari va qutblari o‘zaro bog‘liq bo‘lib ular uchun quyidagi teorema o‘rinli.

12.18-Teorema. z_0 nuqta $f(z)$ funksiyaning qutb maxsus nuqtasi bo‘lishi uchun $\varphi(z)=\frac{1}{f(z)}$ funksiyaning noli bo‘lishi zarur va etarli.

Bu teoremaning isboti cheksiz kichik va cheksiz kattalarning o‘zaro bog‘liqligidan kelib chiqqanligi uchun uni qoldiramiz.

Ravshanki, z_0 nuqta $f(z)$ funksiyaning $n(n\geq 1)$ tartibli qutb nuqtasi bo‘lsa, $\varphi(z)=\frac{1}{f(z)}$ funksiyaning n tartibli noli bo‘ladi.

3. Funksiyaning z_0 nuqta atrofidagi Loran qatoriga yoyilmasida cheksiz sondagi manfiy darajali hadlar ishtirok etadi, ya’ni

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

bo‘lsa z_0 ga funksiyaning ***muhim maxsus nuqtasi*** deb ataladi.

Bunday holda $z \to z_0$ da funksiya hech qanday limitga ega bo‘lmaydi.

Yakkalangan maxsus nuqta atrofida analitik funksiyani Loran qatoriga yoyishning yuqorida qaralgan uchta holi mumkin bo‘lgan barcha hollarni qamrab oladi va ular maxsus nuqta atrofida analitik funksiyaning o‘zini tutishini aniqlashda asosiy ahamiyatga ega.

Yuqorida olib borilgan tahlillardan yakkalangan maxsus nuqtaga nisbatan ikki xil yondashish mumkinligi kelib chiqadi. Biz funksiyani maxsus nuqta atrofida Loran qatoriga yoyib, so‘ngra bu nuqta atrofida funksiya o‘zini qanday tutishini tahlil qildik. Teskari yondashuv ham mumkin bo‘lib, unga ko‘ra asos sifatida yakkalangan maxsus nuqta atrofida funksiyaning o‘zini tutishi olinadi. Bu holda yuqoridagi 12.16 va 12.17-Teoremalarga teskari teoremani isbotlash mumkin.

Cheksiz uzoqlashgan nuqta $f(z)$ funksiyaning yakkalangan maxsus nuqtasi deb ataladi, agar shunday R mavjud bo‘lsaki, funksiya $|z| > R$ sohada $z = 0$ nuqtadan chekli masofada joylashgan maxsus nuqtaga ega bo‘lmasa. Chekli yakkalangan nuqtalar holidagi kabi, bu holda ham maxsus nuqtalarni sinflarga ajratamiz. Buning uchun $R < |z| < \infty$ xalqada analitik bo‘lgan $f(z)$ funksiyani Loran qatoriga yoyamiz

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Cheksizlik - yakkalangan maxsus nuqta uchun quyidagi ta’riflarni qabul qilamiz. Cheksiz uzoqlashgan nuqtaning atrofi deyilganida $\rho < |z| < \infty$ soha, ya’ni $|z| \leq \rho$ doiraning tashqarisi tushuniladi.

12.24-Ta’rif. Agar $f(z)$ funksiyaning cheksiz uzoqlashgan nuqta atrofidagi Loran qatoriga yoyilmasida z ning musbat darajalari ishtirok etmasa:

$$f(z) = \sum_{n=-\infty}^{0} c_n z^n,$$

cheksizlik - yakkalangan maxsus nuqta bu funksiyaning tuzatib bo‘ladigan maxsus nuqtasi deyiladi.

12.25-Ta’rif. Agar (z) funksiyaning cheksiz uzoqlashgan nuqta atrofidagi Loran qatoriga yoyilmasi

$$f(z) = \sum_{n=-\infty}^{k} c_n z^n, \qquad k > 0,\ \ c_k \neq 0$$

ko‘rinishda bo‘lsa, cheksizlik - yakkalangan maxsus nuqta bu funksiyaning qutb maxsus nuqtasi deyiladi,

12.26-Ta’rif. Agar $f(z)$ funksiyaning cheksiz uzoqlashgan nuqta atrofidagi Loran qatoriga yoyilmasi

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

ko‘rinishda bo‘lsa, cheksizlik - yakkalangan maxsus nuqta $f(z)$ funksiyaning muhim maxsus nuqtasi deyiladi,

Cheksizlik $f(z)$ funksiyaning yakkalangan maxsus nuqtasi va $\xi = \frac{1}{z}$ inversiya almashtirishi bo‘lsin. Bu almashtirish 0 ni ∞ ga o‘tkazadi va aksincha. Agar $g(\xi) = f\left(\frac{1}{\xi}\right)$ funksiya uchun $\xi = 0$ tuzatib bo‘ladigan, qutb yoki muhim maxsus nuqta bo‘lsa, u holda cheksizlik $f(z)$ uchun xuddi shunday maxsus nuqta bo‘ladi.

12.21-Misol. $f(z) = \frac{sinz}{z^3}$ funksiyaning maxsus nuqtalari va ularning turini aniqlang.

► Funksiyaning berilishidan ma’lumki, uning bitta $z = 0$ maxsus nuqtasi bo‘lib, u qutb maxsus nuqta bo‘ladi. Qutbning tartibi ikkiga teng, chunki $\sin z$ funksiyaning yoyilmasidan foydalanib yozsak

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} + \cdots\right) =$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots + (-1)^{n-1}\frac{z^{2n-4}}{(2n-1)!} + \cdots$$

Bu berilgan funksiyaning Loran qatori bo‘lib, ta’rifga ko‘ra qutbning tartibi ikki bo‘ladi. ◄

12.22-Misol. $f(z) = \frac{1}{z - \sin z}$ funksiyaning qutblari va ularning turlarini aniqlang.

► Ko‘rinib turibdiki funksiya yagona $z = 0$ maxsus nuqtaga ega bo‘lib, 12.17-Teoremaga asosan u qutb nuqta bo‘ladi, chunki

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{1}{z - \sin z} = \infty$$

Uning tartibini aniqlash uchun $\varphi(z) = z - \sin z$ funksiyani qaraymiz. Bu funksiya uchun

$$\varphi(0) = 0;$$
$$\varphi'(z) = 1 - \cos z, \quad \varphi'(0) = 0;$$
$$\varphi''(z) = -\sin z, \quad \varphi(0) = 0;$$
$$\varphi'''(z) = \cos z, \qquad \varphi'''(0) = 1 \neq 0.$$

Demak, $z = 0$ nuqta $\varphi(z)$ funksiyaning uchinchi tartibli noli bo‘ladi.

12.18-Teoremaga asosan, $z = 0$ nuqta $f(z)$ funksiyaning qutb maxsus nuqtasi bo‘lib, uning tartibi uchga teng. ◄

12.6. Chegirmalar nazariyasi

Funksiyaning maxsus nuqtalardagi chegirmasi va ularni hisoblash. Tadbiqlar uchun muhim bo‘lgan, analitik funksiyalarning chegirmasi tushunchasini kiritamiz

$z = a$ nuqta bir qiymatli $f(z)$ funksiyaning yakkalangan maxsus nuqtasi bo‘lsin. U holda a nuqtani musbat yo‘nalishda bir marta aylanuvchi shunday C oddiy konturni tanlash mumkinki, bu konturda yoki uning ichida a nuqtadan tashqari barcha nuqtalarda $f(z)$ funksiya analitik bo‘ladi.

12.27-Ta’rif. Ushbu

$$\frac{1}{2\pi i}\int_C f(z)dz$$

integral bilan aniqlanuvchi qiymatga $f(z)$ funksiyaning $z = a$ *nuqtadagi chegirmasi* deb ataladi va

$$\text{Res} f(a) = \frac{1}{2\pi i}\int_C f(z)dz \qquad (12.37)$$

ko‘rinishda belgilanadi.

Shuni ta’kidlash lozimki, funksiyaning berilgan nuqtadagi chegirmasi C konturning shakli va o‘lchamlariga bog‘liq emas (agar chiziq oddiy va berilgan nuqtani musbat yo‘nalishda bir marta aylansa).

Ma’lumki, agar a nuqta $f(z)$ funksiyaning yakkalangan maxsus nuqtasi bo‘lsa, funksiyaning a nuqta atrofidagi Loran qatoriga yoyilmasi bosh qismining c_{-1} koffitsienti uchun

$$c_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$$

tenglik o‘rinli. Demak, yuqoridagi ta’rifga asosan

$$\mathrm{Res}f(a) = c_{-1}$$

Agar a nuqta funksiya analitiklik yoki tuzatib bo‘ladigan maxsus nuqtasi bo‘lsa, bu nuqtada chegirma nolga teng. Haqiqatan ham, analitik yoki tuzatib bo‘ladigan a maxsus nuqtada funksiyaning Loran qatoriga yoyilmasida manfiy darajali hadlar qatnashmaydi. Boshqacha aytganda Loran qatorining bosh qismi barcha koeffitsientlari, jumladan c_{-1} ham nolga teng. Shuning uchun

$$c_{-1} = \mathrm{Res}f(a) = 0 \tag{12.38}$$

Agar a nuqta $f(z)$ funksiya uchun qutb yoki muhim maxsus nuqta bo‘lsa, bunday holda chegirma nolga teng bo‘lishi yoki noldan farqli son bo‘lishi ham mumkin.

$z = a$ $(a \neq \infty)$ funksiyaning qutb nuqtasi bo‘lsin. Bu holni ikkiga ajratamiz.

a) oddiy qutb bo‘lgan hol.

Bu holda, ta’rifga ko‘ra $f(z)$ funksiyaning a nuqta atrofidagi Loran qatoriga yoyilmasi

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

ko‘rinishda bo‘ladi. Bu yerdan

$$f(z)(z-a) = c_{-1} + \sum_{n=0}^{\infty} c_n (z-a)^{n+1}$$

yoki $z \to a$ da limitga o‘tsak

$$c_{-1} = \lim_{z\to a} f(z)(z-a).$$

Demak, oddiy qutb nuqta uchun

$$\mathrm{Res}f(a) = \lim_{x\to a} f(z)(z-a) \tag{12.39}$$

Oddiy qutbning quyidagi maxsus ko‘rinishini qaraylik:

$f(z) = \frac{\varphi(z)}{\psi(z)}$, bu yerda $\varphi(z)$ va $\psi(z)$ funksiyalar a nuqtada analitik va $\varphi(a) \neq 0,\ \psi(a) = 0, \psi'(a) \neq 0$, ya’ni a nuqta $f(z)$ funksiya uchun oddiy qutb bo‘ladi. U holda (12.39) formulaga ko‘ra

$$\mathrm{Res}f(a) = \lim_{z\to a} \frac{\varphi(z)}{\psi(z)}(z-a) = \lim_{z\to a} \frac{\varphi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\varphi(a)}{\psi'(a)}$$

Demak bu hol uchun

$$\mathrm{Res}f(a) = \varphi(a)/\,\psi'(a) \tag{12.40}$$

b) Karrali qutb bo‘lgan hol.

Agar a nuqta $f(z)$ funksiyaning m tartibli qutb nuqtasi bo‘lsa, u holda ta’rif bo‘yicha funksiyaning a nuqta atrofida Loran qatori

$$f(z)=\frac{c_{-m}}{(z-a)^m}+\frac{c_{-m+1}}{(z-a)^{m-1}}+\cdots+\frac{c_{-1}}{z-a}+\sum_{n=0}^{\infty}c_n(z-a)^n$$

ko‘rinishda bo‘ladi. Bu tenglikning ikkala tomonini $(z-a)^m$ ga ko‘paytiramiz, u holda

$$f(z)(z-a)^m=c_{-m}+c_{-m+1}(z-a)+$$
$$+\cdots+c_{-1}(z-a)^{m-1}+\sum_{n=0}^{\infty}c_n(z-a)^{n+m}$$

Agar bu tenglikning ikkala tomonini $m-1$ marta differensiallab $z\to a$ da limitga o‘tsak

$$(m-1)!\,c_{-1}=\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[f(z)(z-a)^m]$$

Bu yerdan m tartibli qutb uchun chegirmani hisoblash formulasi

$$\mathrm{Res}f(a)=c_{-1}=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[f(z)(z-a)^m] \qquad (12.41)$$

tenglikni hosil qilamiz.

Xususan, $f(z)=h(z)/(z-a)^m$ va $h(z)$ funksiya $z=a$ nuqtada analitik, hamda $h(a)\neq 0$ bo‘lsa,

$$\mathrm{Res}f(a)=\frac{1}{(m-1)!}h^{(m-1)}(a) \qquad (12.42)$$

12.23-Misol. $f(z)=\frac{z}{(z-1)(z-2)^2}$ funksiyaning maxsus nuqtalaridagi chegirmalarini hisoblang.

► Bu funksiya $z=1$ oddiy qutbga, $z=2$ da ikkinchi tartibli qutbga erishadi. Shuning uchun (12.39) va (12.41) formulalarga ko‘ra

$$\mathrm{Res}f(1)=\lim_{z\to 1}\left[\frac{z}{(z-1)(z-2)^2}(z-1)\right]=\lim_{z\to 1}\left[\frac{z}{(z-2)^2}\right]=1$$

$$\mathrm{Res}f(2)=\frac{1}{(2-1)!}\lim_{z\to 2}\frac{d}{dz}\left[\frac{z}{(z-1)(z-2)^2}(z-2)^2\right]=$$

$$=\lim_{z\to 2}\frac{d}{dz}\left[\frac{z}{(z-1)}\right]=\lim_{z\to 2}\frac{-1}{(z-1)^2}=-1$$ ◄

Muhim maxsus nuqta. Bu holda maxsus nuqtadagi chegirma sifatida funksiyani bevosita bu nuqta atrofida Loran qatoriga yoyib, uning c_{-1} koeffitsienti olinadi.

Cheksiz uzoqlashgan nuqtada chegirma. $f(z)$ funksiya $\rho < |z| < \infty$ sohada, ya'ni $z = \infty$ nuqtaning atrofida analitik, $z = \infty$ nuqta esa funksiyaning yakkalangan maxsus nuqtasi bo'lsin. U holda funksiya $\rho < |z| < \infty$ sohada

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^n + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots$$

kabi Loran qatoriga yoyiladi.

12.28-Ta'rif. $f(z)$ funksiyaning $z = \infty$ nuqtadagi chegirmasi deb,

$$\mathrm{Res} f(\infty) = \frac{1}{2\pi i} \int_{C^-} f(z) dz$$

integral bilan anqlanadigan qiymatga aytiladi, bu yerda C markazi nolda radiusi ρ dan katta ixtiyoriy aylana.

Yuqoridagi chegirmaning ta'rifida integrallash manfiy, ya'ni soat strelkasi yo'nalishida olinadi. Buning sababi integrallashda soha $\rho < |z| < \infty$ chap tomonda qolishi kerak. Demak, cheksiz uzoqlashgan nuqtadagi chegirmani

$$c_{-1} = \frac{1}{2\pi i} \int_{C} f(z) dz$$

integraldan foydalanib topsak,

$$\mathrm{Res} f(\infty) = \frac{1}{2\pi i} \int_{C^-} f(z) dz = -\frac{1}{2\pi i} \int_{C^+} f(z) dz = -c_{-1}$$

tenglikni hosil qilamiz.

Agar ∞ tuzatib bo'ladigan maxsus nuqta bo'lsa, Loran qatori

$$f(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots$$

ko'rinishda bo'ladi. Bu yerdan c_{-1} ni topsak

$$\mathrm{Res} f(\infty) = -\lim_{z \to \infty} z[f(z) - c_0] \qquad (12.43)$$

formula hosil bo'ladi. Demak cheksizlik- tuzatib bo'ladigan maxsus nuqta bo'lsa, u noldan farqli bo'lishi mumkin ekan.

12.24-Misol. $f(z) = 1 + 1/z$ funksiyaning cheksizlikdagi chegirmasini toping.

►Funksiyaning berilishini, uning cheksizlikdagi Loran qatoriga yoyilmasi deb qabul qilsak bo'ladi. Bu funksiya uchun

$$\lim_{z \to \infty} \left(1 + \frac{1}{z}\right) = 1$$

ya'ni $z = \infty$ tuzatib bo'ladigan maxsus nuqta. Shuning uchun $c_{-1} = 1$ ekanligidan $\operatorname{Res} f(\infty) = -1$. ◄

Chegirmalar haqida asosiy teorema. Endi yuqorida kiritilgan tushunchalarning bevosita tadbiqlari bilan shug'ullanamiz. Chegirmalarni amaliy masalalarga tadbiq etishda quyidagi teorema muhim rol o'ynaydi.

12.19-Teorema. (Chegirmalar haqida asosiy teorema) C oddiy kontur bilan chegaralangan D sohaning chekli sondagi $z_1, z_2, \dots, z_n$ nuqtalaridan tashqari barcha nuqtalarida analitik bo'lgan $f(z)$ funksiya berilgan bo'lsin. U holda $\frac{1}{2\pi i}\int\limits_C f(z)dz$ integralning qiymati funksiyaning maxsus nuqtalardagi chegirmalari yig'indisiga teng:

$$\int\limits_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z_k) \tag{12.44}$$

► Barcha maxsus nuqtalarni D sohadan chiqib ketmaydigan qilib $C_1, C_2, \dots, C_n$ aylanalar bilan o'raymiz. Bu aylanalarning ichida faqatgina bitta maxsus nuqta bo'lsin va aylanalardan hech qanday ikkitasi o'zaro kesishmasin. Shunda biz ko'p bog'lamli soha hosil qildik. Agar sohaning chegarasida musbat yo'nalish tanlasak, ichki aylanalarda yo'nalish soat strelkasiga qarama qarshi bo'ladi.(12.16 rasm)

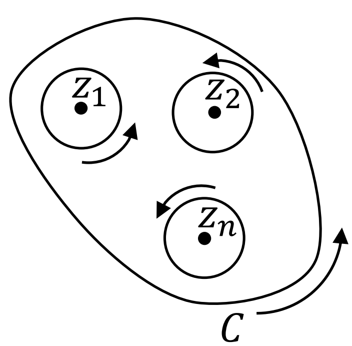
U holda Koshi teoremasining ko'p bog'lamli sohalar uchun natijasiga ko'ra, soha chegarasi bo'yicha olingan integral nolga teng

$$\int\limits_C f(z)dz + \sum_{k=1}^{n} \int\limits_{C_k^-} f(z)dz = 0$$

Agar yig'indida integrallash yo'nalishini o'zgartirsak

$$\int\limits_C f(z)dz = -\sum_{k=1}^{n} \int\limits_{C_k^-} f(z)dz = \sum_{k=1}^{n} \int\limits_{C_k} f(z)dz$$

Bu yerdan (44) tenglik kelib chiqadi. ◄



12.16-rasm

12.20-Teorema. $f(z)$ funksiya kengaytirilgan kompleks tekislikning chekli sondagi $z_1, z_2, \dots, z_n$ va $z = \infty$ nuqtalaridan tashqari barcha nuqtalarda analitik bo‘lsin. U holda funksiyaning barcha nuqtalardagi chegirmalari yig‘indisi nolga teng, ya’ni

$$\sum_{k=1}^{n} \mathrm{Res} f(z_k) + \mathrm{Res} f(\infty) = 0 \qquad (12.45)$$

► $z_1, z_2, \dots, z_n$ nuqtalarni o‘z ichiga oluvchi C yopiq kontur olamiz. U holda yuqoridagi 12.18- Teoremaga ko‘ra

$$\frac{1}{2\pi i} \int_C f(z)dz = \sum_{k=1}^{n} \mathrm{Res} f(z_k)$$

Lekin bu tenglikda chap tomonda turgan integral cheksizlikdagi chegirmaning qarama qarshi ishora bilan olinganiga teng, ya’ni

$$\frac{1}{2\pi i} \int_C f(z)dz = \frac{1}{2\pi i} \int_{C^-} f(z)dz = -\mathrm{Res} f(\infty)$$

Agar integralni chegirmaning qiymati bilan almashtirsak (12.45) tengik kelib chiqadi. ◄

Isbotlangan teorema tasdig‘ining ko‘rinishini biroz o‘zgartirish juda ko‘p integrallarning hisoblanishini osonlashtiradi. Aytaylik, C yopiq kontur berilgan bo‘lib, bu chiziqning ichida n ta maxsus nuqta, tashqi qismida m ta chekli va cheksizlikdagi maxsus nuqtalar yotibdi. U holda (12.45) tenglikka ko‘ra

$$\sum_{k=1}^{n} \mathrm{Res} f(z_k) + \sum_{i=1}^{m} \mathrm{Res} f(z_i) + \mathrm{Res} f(\infty) = 0$$

Ikkinchi tomondan chegirmalar haqidagi asosiy teoremaga ko‘ra

$$\frac{1}{2\pi i} \int_C f(z)dz = \sum_{k=1}^{n} \mathrm{Res} f(z_k)$$

Agar yig‘indini integral bilan almashtirsak

$$\int_C f(z)dz = -2\pi i \sum_{i=1}^{m} \mathrm{Res} f(z_i) - 2\pi i \mathrm{Res} f(\infty) \qquad (12.46)$$

tenglikni hosil qilamiz. Bu tenglikning ma’nosi quyidagicha: integralning qiymati integrallash chizig‘idan tashqaridagi va cheksizlikdagi chegirmalar yig‘indisining qarama qarshi ishora bilan olinganiga teng.

Chegirmalarning aniq integrallarni hisoblashga qo‘llanilishi.

1) **Ratsional funksiyalarni integrallash**

$f(x)=\frac{P_m(x)}{Q_n(x)}$ ratsional funksiya bo‘lib, $P_m(x)$, $Q_n(x)$ mos ravishda $m-$ hamda $n-$ tartibli ko‘phadlar bo‘lsin. Agar $f(x)$ sonlar o‘qining barcha nuqtalarida uzluksiz $Q_n(x)\neq 0$ va $n\geq m+2$, ya’ni maxrajning darajasi suratning darajasidan hech bo‘lmaganda 2 ga ko‘p bo‘lsa, u holda

$$\int_{-\infty}^{\infty} f(x)dx=2\pi i\sigma \qquad (12.47)$$

bu yerda σ soni $f(x)=P_m(x)/Q_n(x)$ funksiyaning yuqori yarim tekislikdagi barcha qutblaridagi chegirmalari yig‘indisini anglatadi.

12.25-Misol. Integralni hisoblang

$$I=\int_0^{\infty}\frac{x^2dx}{(x^2+a^2)^2}, \qquad (a>0)$$

► Integral ostidagi funksiya juft, shuning uchun

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^2}dx$$

$f(z)=\frac{z^2}{(z^2+a^2)^2}$ funksiyani kiritamiz, u haqiqiy o‘qda, ya’ni $z=x$ da integral ostidagi funksiya bilan ustma-ust tushadi.

$f(z)$ funksiya ikkita $z=\pm ai$ ikkinchi tartibli qutbga ega. Yuqori yarim tekislikda bitta $z=ai$ qutb yotganligi uchun

$$\mathrm{Res}f(ai)=\lim_{z\to ai}\frac{d}{dz}[f(z)(z-ai)^2]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{z^2}{(z+ai)^2}\right]=$$

$$=\lim_{z\to ai}\frac{2z(z+ai)^2-2z^2(z+ai)}{(z+ai)^4}=\lim_{z\to ai}\frac{2aiz}{(z+ai)^3}=\frac{1}{4ai}$$

(12.47) tenglikga asosan

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^2}dx=\frac{1}{2}\cdot 2\pi i\cdot\frac{1}{4ai}=\frac{\pi}{4a}$$

qiymatga teng. ◄

2) $\int_0^{\infty} R(x)\cos\lambda x dx$, $\int_0^{\infty} R(x)\sin\lambda x dx$ **ko‘rinishdagi integrallar**

bu yerda $R(x)$ – to‘g‘ri ratsional kasr, $\lambda>0$ ixtiyoriy haqiqiy son.

Bu ko‘rinishdagi integrallarni hisoblash uchun quyidagi lemmadan foydalanish qulay.

Jordan lemmasi. $g(z)$ chekli sondagi nuqtalardan tashqari yuqori yarim tekislikning barcha nuqtalarida analitik funksiya $(0 < \arg z < \pi)$ bo‘lib, $|z| \to \infty$ da bu yarim tekislikda nolga intilsin. U holda $\lambda > 0$ da

$$\lim_{R\to\infty} \int_{C_R} g(z) \cdot e^{i\lambda z} dz = 0,$$

bu yerda C_R markazi O nuqtada, radiusi R ga teng bo‘lgan yuqori yarim tekislikdagi yarim aylana.

12.26-Misol. Integralni hisoblang

$$I = \int_{-\infty}^{0} \frac{x \sin ax}{x^2 + k^2} dx, \quad (a > 0, k > 0)$$

► $f(z) = ze^{iaz}/(z^2 + k^2)$ yordamchi funksiyani kiritamiz. Agar $z = x$ bo‘lsa, $\mathrm{Im} f(z)$ integral ostidagi $\varphi(x) = x \sin ax /(x^2 + k^2)$ funksiya bilan ustma-ust tushadi. C_R konturni qaraymiz. yetarlicha katta R larda $g(z) = \frac{z}{z^2+k^2}$ funksiya

$$|g(z)| \leq \frac{R}{R^2 + k^2}$$

tengsizlikni qanoatlantiradi va o‘z navbatida $R \to \infty$ da $g(z) \to 0$. Demak Jordan lemmasiga ko‘ra

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iaz}}{z^2 + k^2} dz = 0$$

Ixtiyoriy $R > k$ uchun chegirmalar haqidagi teoremaga asosan

$$\int_{-R}^{R} \frac{xe^{iax}}{x^2 + k^2} dx + \int_{C_R} \frac{ze^{iaz}}{z^2 + k^2} dz = 2\pi i \sigma$$

bu yerda σ soni $f(z) = \frac{ze^{iaz}}{z^2+k^2}$ funksiyaning yuqori yarim tekislikdagi chegirmasi

$$\sigma = \mathrm{Res} f(ik) = \lim_{z\to ik} \left[\frac{ze^{iaz}}{z^2 + k^2} \cdot (z - ik) \right] = \frac{1}{2} e^{-ak}$$

$R \to \infty$ da

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2 + k^2} dx = \pi i e^{-ak}$$

Chap va o‘ng tomondan haqiqiy va mavhum qismlarini ajratsak

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2 + k^2} dx = \pi e^{-ak}$$

integralning qiymatini topamiz. Integral ostidagi funksiya juft bo‘lganligi uchun

$$I=\frac{\pi}{2}e^{-ak}$$

berilgan integral qiymati hosil bo‘ladi. ◄

3) $I=\int\limits_0^{2\pi} R(\cos\varphi,\sin\varphi)d\varphi$ **kop’rinishdagi integrallar**

Bu yerda R argumentlariga nisbatan ratsional funksiya. Integralda $z=e^{i\varphi}$ almashtirish bajaramiz, u holda

$$\sin\varphi=\frac{1}{2i}\left(z-\frac{1}{z}\right),\cos\varphi=\frac{1}{2}\left(z+\frac{1}{z}\right),\qquad d\varphi=-i\frac{dz}{z}$$

$0\le\varphi\le 2\pi$ kabi bo‘lganida z o‘zgaruvchi $|z|=1$ aylanani musbat yo‘nalishda aylanib chiqadi. Shuning uchun yuqoridagi integral

$$\int\limits_{|z|=1} R_1(z)dz$$

ko‘rinishdagi yopiq kontur bo‘yicha integralga keladi, bu yerda

$$R_1(z)=-\frac{i}{z}\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)$$

Chegirmalar haqidagi teoremaga asosan

$$I=2\pi i\sum_{k=1}^{n}\mathrm{Res}f(z_k)$$

bu yerda $z_1,z_2,\dots,z_n$ lar $R_1(z)$ ratsional funksiyaning $|z|\le 1$ doiradagi qutblari.

12.27-Misol. Quyidagi integralni hisoblang

$$I=\int\limits_0^{2\pi}\frac{dx}{7+3\cos x}$$

► $z=e^{ix}$, $\cos x=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$ almashtirish bajaramiz. U holda $dx=-i\frac{dz}{z}$, $|z|=1$. Integralni yangi o‘zgaruvchida shakl almashtiramiz

$$I=-i\int\limits_{|z|=1}\frac{dz}{z\left(7+3\frac{z^2+1}{2z}\right)}=-2i\int\limits_{|z|=1}\frac{dz}{3z^2+14z+3}$$

$$=-2i\cdot 2\pi i\sum_{k=1}^{n}\mathrm{Res}f(z_k)$$

Integral ostidagi funksiyaning qutblari

$$z_1=-\frac{7+2\sqrt{10}}{3},\qquad z_2=-\frac{7-2\sqrt{10}}{3}$$
bo‘lib ulardani faqat z_2 qutb $|z|=1$ aylananing ichiga tushadi. Bu nuqtadagi chegirmani hisoblaymiz

$$\mathrm{Res}f(z_2)=$$
$$=\lim_{z\to z_2}\frac{1}{3\left(z+\frac{7+2\sqrt{10}}{3}\right)\left(z-\frac{2\sqrt{10}-7}{3}\right)}\left(z-\frac{2\sqrt{10}-7}{3}\right)=$$
$$=\lim_{z\to z_2}\frac{1}{3\left(z+\frac{7+2\sqrt{10}}{3}\right)}=\frac{1}{4\sqrt{10}}$$

Demak,
$$I=-2i\cdot 2\pi i\frac{1}{4\sqrt{10}}=\frac{\pi}{\sqrt{10}}$$
bo‘ladi.◄

Nazorat savollari

1. Kompleks tekislikda ochiq doira deb nimaga aytiladi?
2. Ko‘rsatgichli funksiyaning ta’rifini keltiring.
3. Logarifmik funksiyaning ta’rifini keltiring.
4. Koshi- Riman shartlari deb nimaga aytiladi?
5. Qanday funksiyalar qo‘shma garmonik deyiladi?
6. Konform akslantirishni ta’riflab bering.
7. Yopiq kontur uchun Koshi teoremasini ayting.
8. Koshining integral formulasini yozing.
9. Abel teoremasini aytib bering.
10. Loran qatorini yozib bering.
11. Yakkalangan maxsus nuqta turlarini ta’riflang.
12. Kompleks funksiya chegirmasi deb nimaga aytiladi?

Mashqlar

1. $|z+2|<|z-3i|$ tengsizlikni qanoatlantiruvchi nuqtalar to‘plamini aniqlang.

2. $|z|\mathrm{Re}z=5$ tenglikni qanoatlantiruvchi nuqtalar to‘plami tekislikda qanday chiziqni aniqlaydi?

3. $w = (z - i)/(z + i)$ funksiyaning haqiqiy va mavhum qismlarini ajrating.

4. $w = \sin z$ funksiyaning $z = \pi + i\ln(2 + \sqrt{5})$ nuqtadagi qiymatini toping.

Quyidagi funksiyalarni analitiklikka tekshiring

5. $w = z^3 + z$, **6.** $w = z + \mathrm{Im} z$.

Quyidagi shartlar bo‘yicha analitik $f(z)$ funksiyani toping.

7. $u(x, y) = x^3 - 3xy^2, f(0) = 0$. **8.** $v(x, y) = 4x^3y - 4y^3x$, $f(0) = 0$. **9.** $u(x, y) = x/(x^2 + y^2)$, $f(2) = 1/2$.

Integrallarni hisoblang

10. $\int\limits_{|z|=2} \frac{dz}{z^2-4z+3}$. **11.** $\int\limits_{|z-i|=2} \frac{dz}{z^2+3z}$. **12.** $\int\limits_{|z-1|=2} \frac{e^z dz}{(z^2-4)^2}$.

13. $\int\limits_{|z-i+1|=2} \frac{dz}{z(z+3)(z^2-1)}$.

Funksiyalarni ko‘rsatilgan xalqalarda Loran qatoriga yoying:

14. $f(z) = \frac{1}{z^2-7z+10}$, $2 < |z| < 3$.

15. $f(z) = \frac{1}{(z^2+2z-3)^2}$, $0 < |z + 1| < 2$.

Funksiyaning maxsus nuqtalarini topib, turlarini aniqlang va bu nuqtalardagi chegirmalarni hisoblang

16. $f(z) = \frac{e^z}{z^2+z-2}$. **17.** $f(z) = \frac{\sin\frac{1}{z}}{1-z}$. **18.** $f(z) = \frac{z}{1-\cos z}$.

Quyidagi integrallarni Koshining chegirmalar haqidagi teoremasi yordamida hisoblang

19. $\int\limits_{|z|=4} \frac{(z+1)dz}{z^2+2z-3}$. **20.** $\int\limits_{|z-1|=1} \frac{\sin \pi z dz}{(z^2-1)^2}$.

Javoblar

1. $\left\{z = x + iy : y < -\frac{1}{3}x + \frac{5}{6}\right\}$; **2.** Nuqtalarining koordinatalari $x^2 + y^2 = \frac{25}{x^2}$ tenglikni qanoatlanuvchi chiziqni aniqlaydi; **3.** $u(x, y) = \frac{x^2+y^2-1}{x^2+(y+1)^2}, v(x, y) = -\frac{2x}{x^2+(y+1)^2}$; **4.** $-2i$; **5.** Analitik; **6.** Birorta nuqtada ham analitik emas; **7.** $f(z) = z^3$; **8.** $f(z) = z^4$; **9.** $f(z) = 1/z$; **10.** $-i\pi$; **11.** $2\pi i/3$; **12.** $e^2\pi i/16$; **13.** $-1/12$; **14.** $-\frac{1}{3}\sum\limits_{n=0}^{\infty}\left(\frac{1}{5^{n+1}} + +\frac{1}{2^{n+1}}\right)z^n$; **15.** $\frac{2}{(z-1)^2} -$

$\frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^n \frac{2n+3}{4^{n+1}} (z-1)^n$; **16.** $z = -2$ va $z = 1$ oddiy qutb nuqtalari; **17.** $z = 1$ oddiy qutb va $z = 0$ muhim maxsus nuqta; **18.** $z = 0$ tuzatib bo‘ladigan maxsus nuqta; **19.** $2\pi i$; **20.** $-\pi^2 i/2$;

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